

Sequential Joint Detection and Estimation: Optimum Tests and Applications

Yasin Yilmaz, Shang Li, and Xiaodong Wang

Abstract

We treat the statistical inference problems in which one needs to detect the correct signal model among multiple hypotheses and estimate a parameter simultaneously using as small number of samples as possible. Conventional methods treat the detection and estimation subproblems separately, ignoring the intrinsic coupling between them. However, a joint detection and estimation problem should be solved to maximize the overall performance. We address the sample size concern through a sequential and Bayesian setup. Specifically, we seek the optimum triplet of stopping time, detector, and estimator(s) that minimizes the number of samples subject to a constraint on the combined detection and estimation cost. A general framework for optimum sequential joint detection and estimation is developed. The resulting optimum detector and estimator(s) are strongly coupled with each other, proving that the separate treatment is strictly sub-optimum. The theoretical results derived for a quite general model are then applied to several problems with linear quadratic Gaussian (LQG) models, including dynamic spectrum access in cognitive radio, and state estimation in smart grid with topological uncertainty. Numerical results corroborate the superior overall detection and estimation performance of the proposed schemes over the conventional methods that handle the subproblems separately.

Index Terms

joint detection and estimation, sequential methods, stopping time, dynamic spectrum access, state estimation with topological uncertainty

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I. INTRODUCTION

Detection and estimation problems appear simultaneously in a wide range of fields, such as wireless communications, power systems, image processing, genetics, and finance. For instance, to achieve effective and reliable dynamic spectrum access in a cognitive radio system, a secondary user needs to detect primary user transmissions, and if detected to estimate the cross channels that may cause interference to primary users [1]. In power grid monitoring, it is essential to detect the correct topological model, and at the same time estimate the system state [2]. Some other important examples are detecting and estimating objects from images [3], target detection and parameter estimation in radar [4], and detection and estimation of periodicities in DNA sequences [5].

In all these applications, detection and estimation problems are intrinsically coupled, and are both of primary importance. Hence, a jointly optimum method, that maximizes the overall performance, is needed. Classical approaches either treat the two subproblems separately with the corresponding optimum solutions, or solve them together, as a composite hypothesis testing problem, using the generalized likelihood ratio test (GLRT) or its alternatives such as the Rao test, the Wald test, the Durbin test, and the Terrell test [6]. However, such approaches do not yield the overall optimum solution [8], [9]. In the former approach, for example, the likelihood ratio test is performed by averaging over the unknown parameters to solve the detection subproblem optimally; and then based on the detection decision, the Bayesian estimators are used to solve the estimation subproblem. On the other hand, in GLRT, the maximum likelihood (ML) estimates of all unknown parameters are computed, and then using these estimates, the likelihood ratio test is performed as in a simple hypothesis testing problem. In GLRT, the primary emphasis is on the detection performance and the estimation performance is of secondary importance. GLRT is very popular due to its simplicity and asymptotic optimality [7]. However, even its detection performance is not optimal in the Neyman-Pearson sense [10], and neither is the overall performance under mixed Bayesian/Neyman-Pearson [11] and pure Bayesian [8] setups.

The first systematic theory on joint detection and estimation appeared in [8]. This initial work, in a Bayesian framework, derives optimum joint detector and estimator structures for different levels of coupling between the two subproblems. [12] extends the results of [8] on binary hypothesis testing to the multi-hypothesis case. In [13], different from [8], [12], the case with unknown parameters under the null hypothesis is considered. [13] does not present an optimum joint detector and estimator, but shows that, even in the classical separate treatment of the two subproblems, likelihood ratio test implicitly uses the posterior distributions of unknown parameters, which characterize the Bayesian estimation. [14] deals

with joint multi-hypothesis testing and non-Bayesian estimation considering a finite discrete parameter set and the minimax approach. [11], [15]–[17] study Bayesian estimation under different Neyman-Pearson-like formulations, and derive the corresponding optimum joint detection and estimation schemes. [5], in a minimax sense, extends the analysis in [15] to the general case with unknown parameters in both hypotheses. [2] handles the joint multi-hypothesis testing and state estimation problem for linear models with Gaussian noise. It finds the joint posterior distribution of the hypotheses and the system states, which can be used to identify the optimum joint detector and estimator for a specific performance criterion in a unified Bayesian approach.

Most of the today’s engineering applications are subject to resource (e.g., time, energy, bandwidth) constraints. For that reason, it is essential to minimize the number of observations used to perform a task (e.g., detection, estimation) due to the cost of taking a new observation, and also latency constraints. Sequential statistical methods are designed to minimize the average number of observations for a given accuracy level. They are equipped with a stopping rule to achieve optimal stopping, unlike fixed-sample-size methods. Specifically, we cannot stop taking samples too early due to the performance constraints, and do not want to stop too late to save critical resources, such as time and energy. Optimal stopping theory handles this trade-off through sequential methods. For more information on sequential methods we refer to the original work [18] by Wald, and a more recent book [19]. The majority of existing works on joint detection and estimation consider only the fixed-sample-size problem. Although [13] discusses the case where observations are taken sequentially, it does not consider optimal stopping, limiting the scope of the work to the iterative computation of sufficient statistics. The only work that treats the joint detection and estimation problem in a “real” sequential manner is [9]. It provides the exact optimum triplet of stopping time, detector, and estimator for a linear scalar observation model with Gaussian noise, where there is an unknown parameter only under the alternative hypothesis.

In this paper, we solve the optimum sequential joint detection and estimation problem under the most general setup, namely for a general non-linear vector signal model with arbitrary noise distribution and unknown parameters under multiple hypotheses. We also do not assume a specific estimation cost function.

The remainder of the paper is organized as follows. In Section II, we derive the optimum procedure for sequential joint detection and estimation under a general setup. We then apply the theory developed in Section II to a general linear quadratic Gaussian model in Section III, dynamic spectrum access in cognitive radio networks in Section IV, and state estimation in smart grid with topological uncertainty in Section V. Finally, concluding remarks are given in Section VI.

II. OPTIMUM SEQUENTIAL JOINT DETECTION AND ESTIMATION

A. Problem Formulation

Consider a general model

$$\mathbf{y}_t = f(\mathbf{x}, \mathbf{H}_t) + \mathbf{w}_t, \quad t = 1, 2, \dots, \quad (1)$$

where $\mathbf{y}_t \in \mathbb{R}^M$ is the measurement vector taken at time t ; $\mathbf{x} \in \mathbb{R}^N$ is the unknown vector of parameters that we want to estimate; \mathbf{H}_t is the observation matrix that relates \mathbf{x} to \mathbf{y}_t ; f is a (possibly nonlinear) function of \mathbf{x} and \mathbf{H}_t ; and $\mathbf{w}_t \in \mathbb{R}^M$ is the noise vector.

In addition to estimation, we would like to detect the true hypothesis (H_i , $i = 0, 1, \dots, I$) in a multiple hypothesis testing setup, in which \mathbf{x} is distributed according to a specific probability distribution under each hypothesis, i.e.,

$$H_i : \mathbf{x} \sim \pi_i, \quad i = 0, 1, \dots, I. \quad (2)$$

We assume that the two processes $\{\mathbf{w}_t\}$, $\{\mathbf{H}_t\}$ are independent and independent from the random vector \mathbf{x} . The function f , which is possibly nonlinear, represents the dependencies between \mathbf{y}_t and \mathbf{H}_t , and \mathbf{y}_t and \mathbf{x} . For generality we do not specify f , and also the probability distributions for \mathbf{x} , \mathbf{H}_t , \mathbf{w}_t . We also assume that $\{\mathbf{y}_t, \mathbf{H}_t\}$ are observed at each time t , and we know the probability distributions $\{\pi_i\}$, the probability density function $p(\{\mathbf{w}_t\})$, and the link function f . Note that we allow for correlated noise \mathbf{w}_t and correlated \mathbf{H}_t , both temporally and spatially, as well as correlated \mathbf{x} . Note also that observable random \mathbf{H}_t is a more general model than deterministic and known \mathbf{H}_t . In (2), there is no assumption on \mathbf{H}_t ; hence, different hypotheses may lead to quite different \mathbf{H}_t observations. The linear model $\mathbf{y}_t = \mathbf{H}_t \mathbf{x} + \mathbf{w}_t$ is commonly used in many applications. For example, in system identification, \mathbf{x} is the unknown system coefficients, \mathbf{H}_t is the (random) input applied to the system, and \mathbf{y}_t is the output at time t . Another example is the estimation of wireless channel coefficients, in which \mathbf{x} is the unknown channel coefficients, \mathbf{H}_t is the transmitted (random) pilot signal, \mathbf{y}_t is the received signal, and \mathbf{w}_t is the additive channel noise.

We denote with \mathcal{F}_t and $\{\mathcal{F}_t\}$ the sigma-algebra and filtration generated by the history of the observation matrices $\{\mathbf{H}_1, \dots, \mathbf{H}_t\}$, respectively. Similarly, \mathcal{G}_t and $\{\mathcal{G}_t\}$ denote the sigma-algebra and filtration generated by $\{(\mathbf{y}_1, \mathbf{H}_1), \dots, (\mathbf{y}_t, \mathbf{H}_t)\}$. Since we want to both detect and estimate, similar to [8], [11], [12], we use a combined cost function

$$\mathcal{C}(T, d_T, \{\hat{\mathbf{x}}_T^i\}) = \sum_{i=0}^I a_i P_i(d_T \neq i | \mathcal{F}_T) + \sum_{i=0}^I \sum_{j=0}^I b_{ij} E_i \left[J(\hat{\mathbf{x}}_T^j, \mathbf{x}) \mathbb{1}_{\{d_T=j\}} | \mathcal{F}_T \right] \quad (3)$$

where T is the stopping time, d_T is the detection function, $\hat{\mathbf{x}}_T^j$ is the estimator when we decide on H_j , $J(\hat{\mathbf{x}}_T, \mathbf{x})$ is a general estimation cost function, e.g., $\|\hat{\mathbf{x}}_T - \mathbf{x}\|^2$, P_i and E_i are the probability measure and expectation under H_i , and $\{a_i, b_{ij}\}_{i,j=0,\dots,I}$ are some constants. The estimation cost function $J(\hat{\mathbf{x}}_t, \mathbf{x})$ is assumed known, but not specified for generality. The indicator function $\mathbb{1}_{\{A\}}$ takes the value 1 if the event A is true, or 0 otherwise. In (3), the first summation is the detection cost, and the remaining ones are the estimation cost. Writing (3) in the following alternative form

$$\mathcal{C}(T, d_T, \{\hat{\mathbf{x}}_T^i\}) = \sum_{i=0}^I E_i \left[b_{ii} J(\hat{\mathbf{x}}_T^i, \mathbf{x}) \mathbb{1}_{\{d_T=i\}} + \left\{ a_i + \sum_{j=0, j \neq i}^I b_{ij} J(\hat{\mathbf{x}}_T^j, \mathbf{x}) \mathbb{1}_{\{d_T=j\}} \right\} \mathbb{1}_{\{d_T \neq i\}} \middle| \mathcal{F}_T \right] \quad (4)$$

it is clear that our cost function corresponds to the Bayes risk given $\{\mathbf{H}_1, \dots, \mathbf{H}_t\}$.

The combined cost in (3) can also be seen as a *scalarization* of multiple detection and estimation costs, which gives a *Pareto optimal* solution [20]. It is natural to combine the detection and estimation costs since they both penalize the wrong decision event. Specifically, as can be seen in the curly brackets in (4), the wrong decision event is penalized with the constant a_i by the detection cost; and with a cost that depends on the estimator $\hat{\mathbf{x}}_T^j$, the real parameter values \mathbf{x} , and the decision d_T by the estimation cost. Hence, as in [8, Section IV], [11, Section III], and [12, Section II], combining the detection and estimation costs we define a Bayesian cost for the overall problem that involves detection and estimation. Note that the detection and estimation costs have different ranges: detection error probabilities are restricted to $[0, 1]$, whereas the estimation cost (e.g., mean squared error) could be any nonnegative real number. An upper bound η for the estimation cost, which can be estimated through training data or determined using the domain knowledge about the unknown parameter vector \mathbf{x} , can be used, as in [21, eq. (22)], to normalize the estimation cost to $[0, 1]$, i.e., $b_{ij} = \frac{c_{ij}}{\eta}$, $\forall i, j$. Then, the choice of $\{a_i, c_{ij}\}$ strikes a balance between the detection and estimation costs. If it is desired to satisfy a given set of separate constraints on the detection and estimation costs, a feasible point can be searched on a grid composed of a discrete set of $\{a_i, c_{ij}\}$ values, similar to the numerical computation of Lagrange multipliers, e.g., [22, Algorithm 3], [23, Algorithm 4].

In a sequential setup, in general, the expected stopping time (i.e., the average number of samples) is minimized subject to a constraint on the cost function. In the presence of an *ancillary statistic*, such as \mathcal{F}_T , that does not depend on the unknown parameters \mathbf{x} , conditioning is known to have significant advantages [24], hence the cost function in (3) is conditioned on \mathcal{F}_T . Intuitively, there is no need to average the performance measure $\mathcal{C}(T, d_T, \{\hat{\mathbf{x}}_T^i\})$ over \mathcal{F}_T , which is an observed statistic. This is known as the *conditionality principle* [25]. Conditioning on \mathcal{F}_T also frees our formulation from assuming statistical

descriptions (e.g., probability distribution, independence, stationarity) on the observation matrices $\{\mathbf{H}_t\}$.

From an operational point of view, we start with the following stopping rule: stop the first time the target accuracy level α is achieved, i.e., the inequality $\mathcal{C}(T, d_T, \{\hat{\mathbf{x}}_T^i\}) \leq \alpha$ is satisfied. This operational problem statement gives us the problem formulation

$$\min_{T, d_T, \{\hat{\mathbf{x}}_T^i\}} T \quad \text{subject to} \quad \mathcal{C}(T, d_T, \{\hat{\mathbf{x}}_T^i\}) \leq \alpha, \quad (5)$$

which in turn defines an $\{\mathcal{F}_t\}$ -adapted stopping time T . This is because T is solely determined by $\mathcal{C}(T, d_T, \{\hat{\mathbf{x}}_T^i\})$, which, as seen in (3), averages over $\{\mathbf{y}_t\}$ and thus is a function of only $\{\mathbf{H}_t\}$. The stopping rule considered here is a natural extension of the one commonly used in sequential estimation problems, e.g., [28], [30], and is optimum for $\{\mathcal{F}_t\}$ -adapted stopping times, as shown in (5). Note that the solution sought in (5) is optimum for each realization of $\{\mathbf{H}_t\}$, and not on average with respect to this sequence.

Recall that \mathcal{G}_t and $\{\mathcal{G}_t\}$ denote the sigma-algebra and filtration generated by the complete history of observations $\{(\mathbf{y}_1, \mathbf{H}_1), \dots, (\mathbf{y}_t, \mathbf{H}_t)\}$, respectively, thus $\mathcal{F}_t \subset \mathcal{G}_t$. In the pure detection and pure estimation problems, it is well known that serious analytical complications arise if we consider a general $\{\mathcal{G}_t\}$ -adapted stopping time, that depends on the complete history of observations. Specifically, in the pure estimation problem, finding the optimum sequential estimator that attains the sequential Cramer-Rao lower bound (CRLB) is not a tractable problem if T is adapted to the complete observation history $\{\mathcal{G}_t\}$ [26], [27]. Similarly, in the pure detection problem with an $\{\mathcal{G}_t\}$ -adapted stopping time, we end up with a two-dimensional optimal stopping problem which is impossible to solve (analytically) since the thresholds for the running likelihood ratio depend on the sequence $\{\mathbf{H}_t\}$. Alternatively, in [9], [28]–[30], T is restricted to $\{\mathcal{F}_t\}$ -adapted stopping times, which facilitates obtaining an optimal solution. In this paper, we are interested in $\{\mathcal{F}_t\}$ -adapted stopping times as well. Hence, $E[T|\mathcal{F}_T] = T$ and we aim to solve the optimization problem in (5).

B. Optimum Solution

Optimum Estimators: Let us begin our analysis with the optimum estimators.

Lemma 1. *For the problem in (5), the optimum estimator $\hat{\mathbf{x}}_T^i$ when H_i is decided is given by*

$$\hat{\mathbf{x}}_T^i = \arg \min_{\hat{\mathbf{x}}} \bar{E}_i [J(\hat{\mathbf{x}}, \mathbf{x}) | \mathcal{G}_T], \quad i = 0, 1, \dots, I, \quad (6)$$

where \bar{E}_i is the expectation under the probability distribution

$$\bar{p}_T^i(\mathbf{x}|\mathcal{G}_T) \triangleq \frac{\sum_{j=0}^I b_{ji} L_T^{ji} p_j(\mathbf{x}|\mathcal{G}_T)}{\sum_{j=0}^I b_{ji} L_T^{ji}}, \quad (7)$$

$$\left(\text{e.g., } \bar{p}_T^i(\mathbf{x}|\mathcal{G}_T) = \frac{b_{0i} p_0(\mathbf{x}|\mathcal{G}_T) + b_{1i} L_T^{10} p_1(\mathbf{x}|\mathcal{G}_T)}{b_{0i} + b_{1i} L_T^{10}} \text{ for } I = 2 \right), \quad (8)$$

$p_j(\mathbf{x}|\mathcal{G}_T)$ is the posterior distribution under H_j , and

$$L_T^{ji} \triangleq \frac{p_j(\{\mathbf{y}_t\}_{t=1}^T | \mathcal{F}_T)}{p_i(\{\mathbf{y}_t\}_{t=1}^T | \mathcal{F}_T)} \quad (9)$$

is a likelihood ratio. Specifically, the minimum mean-squared error (MMSE) estimator, for which

$J(\hat{\mathbf{x}}, \mathbf{x}) = \|\hat{\mathbf{x}} - \mathbf{x}\|^2$, is given by

$$\hat{\mathbf{x}}_T^i = \frac{\sum_{j=0}^I b_{ji} L_T^{ji} E_j[\mathbf{x}|\mathcal{G}_T]}{\sum_{j=0}^I b_{ji} L_T^{ji}} \quad (10)$$

$$\left(\text{e.g., } \hat{\mathbf{x}}_T^i = \frac{b_{0i} E_0[\mathbf{x}|\mathcal{G}_T] + b_{1i} L_T^{10} E_1[\mathbf{x}|\mathcal{G}_T]}{b_{0i} + b_{1i} L_T^{10}} \text{ for } I = 2 \right). \quad (11)$$

Proof: See Appendix A. ■

We see that the MMSE estimator in (10) is the weighted average of the MMSE estimators under $\{H_i\}$. Note that typically the likelihood ratios $\{L_T^{ji}\}_{j \neq i}$ is smaller than 1 under H_i , that is, $\hat{\mathbf{x}}_T^i$ is close to $E_i[\mathbf{x}|\mathcal{G}_T]$.

With the optimum estimators given in (6) the cost function in (3) becomes

$$\mathcal{C}(T, d_T, \{\hat{\mathbf{x}}_T^i\}) = \sum_{i=0}^I a_i P_i(d_T \neq i | \mathcal{F}_T) + \sum_{i=0}^I \sum_{j=0}^I b_{ij} E_i \left[\underbrace{E_i \left[J(\hat{\mathbf{x}}_T^j, \mathbf{x}) | \mathcal{G}_T \right]}_{\Delta_T^{ij}} \mathbb{1}_{\{d_T=j\}} | \mathcal{F}_T \right] \quad (12)$$

where Δ_T^{ij} is the posterior expected estimation cost when H_j is decided under H_i .

Specifically, for the MMSE estimator

$$\begin{aligned} \Delta_T^{ij} &= E_i \left[\|\mathbf{x} - \hat{\mathbf{x}}_T^j\|^2 | \mathcal{G}_T \right] = E_i \left[\|\mathbf{x} - E_i[\mathbf{x}|\mathcal{G}_T] + E_i[\mathbf{x}|\mathcal{G}_T] - \hat{\mathbf{x}}_T^j\|^2 | \mathcal{G}_T \right] \\ &= E_i \left[\|\mathbf{x} - E_i[\mathbf{x}|\mathcal{G}_T]\|^2 | \mathcal{G}_T \right] + E_i \left[\|E_i[\mathbf{x}|\mathcal{G}_T] - \hat{\mathbf{x}}_T^j\|^2 | \mathcal{G}_T \right] - 2E_i \left[(\mathbf{x} - E_i[\mathbf{x}|\mathcal{G}_T])' (E_i[\mathbf{x}|\mathcal{G}_T] - \hat{\mathbf{x}}_T^j) | \mathcal{G}_T \right] \\ &= \text{Tr}(\text{Cov}_i[\mathbf{x}|\mathcal{G}_T]) + \|E_i[\mathbf{x}|\mathcal{G}_T] - \hat{\mathbf{x}}_T^j\|^2, \end{aligned} \quad (13)$$

where $\text{Tr}(\cdot)$ is the trace of a matrix. For binary hypothesis testing, where $I = 2$,

$$\Delta_T^{ij} = \text{Tr}(\text{Cov}_i[\mathbf{x}|\mathcal{G}_T]) + \delta_T^{ij} \|E_0[\mathbf{x}|\mathcal{G}_T] - E_1[\mathbf{x}|\mathcal{G}_T]\|^2, \quad (14)$$

$$\delta_T^{0j} = \left(\frac{b_{1j} L_T^{10}}{b_{0j} + b_{1j} L_T^{10}} \right)^2 \quad \text{and} \quad \delta_T^{1j} = \left(\frac{b_{0j}}{b_{0j} + b_{1j} L_T^{10}} \right)^2. \quad (15)$$

We used the fact that $E_i[\mathbf{x}|\mathcal{G}_T]$ and $\hat{\mathbf{x}}_T^j$ are \mathcal{G}_T -measurable, i.e., deterministic given \mathcal{G}_T , to write (13), and the MMSE estimator in (10) to write (14). According to (13), Δ_T^{ij} is the MMSE under H_i plus the

distance between our estimator $\hat{\mathbf{x}}_T^j$ and the optimum estimator under H_i . The latter is the penalty we pay for not knowing the true hypothesis.

Optimum Detector: We now search for the optimum decision function d_T that minimizes (12) for any stopping time T .

Lemma 2. *The optimum detector d_T for the problem in (5) is given by*

$$d_T = \arg \min_i \sum_{j=0, j \neq i}^I \left(a_j + b_{ji} \Delta_T^{ji} - b_{jj} \Delta_T^{jj} \right) p_j(\{\mathbf{y}_t\}_{t=1}^T | \mathcal{F}_T) \quad (16)$$

$$\left(\text{e.g., } d_T = \begin{cases} 1 & \text{if } L_T (a_1 + b_{10} \Delta_T^{10} - b_{11} \Delta_T^{11}) \geq a_0 + b_{01} \Delta_T^{01} - b_{00} \Delta_T^{00} \\ 0 & \text{otherwise} \end{cases}, \text{ for } I = 2 \right), \quad (17)$$

where $L_T = \frac{p_1(\{\mathbf{y}_t\}_{t=1}^T | \mathcal{F}_T)}{p_0(\{\mathbf{y}_t\}_{t=1}^T | \mathcal{F}_T)}$ is the likelihood ratio, and $\Delta_T^{ij} = \mathbb{E}_i \left[J(\hat{\mathbf{x}}_T^j, \mathbf{x}) | \mathcal{G}_T \right]$ is the posterior expected estimation cost.

Proof: See Appendix B. ■

The optimum decision function d_t is coupled with the estimators $\{\hat{\mathbf{x}}_t^i\}$ through the posterior estimation costs $\{\Delta_t^{ij}\}$ due to our joint formulation [cf. (3)]. Specifically, while making a decision, it takes into account, in a very intuitive way, all possible estimation costs that may result from the true hypothesis and its decision. For example, under H_i small Δ_T^{ii} , which is the estimation cost for deciding on H_i , facilitates choosing H_i . On the other hand, the reverse is true for Δ_T^{ij} , which corresponds to the wrong decision cases. That is, large Δ_T^{ij} favors $d_T = i$. In the detection-only problem with $b_{ij} = 0, \forall i, j$, the coupling disappears, and d_T boils down to the multi-hypothesis version of the well-known likelihood ratio test (i.e., maximum likelihood test or maximum a posteriori test if a_i represents the prior for H_i).

Complete Solution: We can now identify the optimum stopping time T , and as a result the complete solution $(T, d_T, \hat{\mathbf{x}}_T)$ to the optimization problem in (5).

Theorem 1. *The optimum sequential joint detector and estimator $(T, d_T, \hat{\mathbf{x}}_T)$ that solves the problem in*

(5) is given by

$$T = \min\{t \in \mathbb{N} : C_t \leq \alpha\} \quad (18)$$

$$d_T = \arg \min_i \sum_{j=0, j \neq i}^I \left(a_j + b_{ji} \Delta_T^{ji} - b_{jj} \Delta_T^{jj} \right) p_j(\{\mathbf{y}_t\}_{t=1}^T | \mathcal{F}_T) \quad (19)$$

$$\hat{\mathbf{x}}_T = \arg \min_{\hat{\mathbf{x}}} \bar{E}_{d_T} [J(\hat{\mathbf{x}}, \mathbf{x}) | \mathcal{G}_T] \quad (20)$$

$$\left(\text{e.g., } \hat{\mathbf{x}}_T = \frac{\sum_{j=0}^I b_{jd_T} L_T^{jd_T} E_j[\mathbf{x} | \mathcal{G}_T]}{\sum_{j=0}^I b_{jd_T} L_T^{jd_T}} \text{ for } J(\hat{\mathbf{x}}, \mathbf{x}) = \|\hat{\mathbf{x}} - \mathbf{x}\|^2 \right) \quad (21)$$

where

$$C_t \triangleq \sum_{i=0}^I E_i \left[b_{ii} \Delta_t^{ii} \mathbb{1}_{\{d_t=i\}} + \sum_{j=0, j \neq i}^I \left(a_i + b_{ij} \Delta_t^{ij} \right) \mathbb{1}_{\{d_t=j\}} | \mathcal{F}_t \right] \quad (22)$$

is the optimal cost at time t . The probability distribution \bar{p}_t^i for the expectation \bar{E}_i , and the likelihood ratio L_t^{ji} are given in (7) and (9), respectively. For the posterior estimation cost Δ_t^{ij} see (12)–(15).

Proof: In Lemma 1, we showed that $\{\hat{\mathbf{x}}_T^i\}$ minimize the cost function in (3) for any stopping time T and decision function d_T , i.e., $\mathcal{C}(T, d_T, \{\hat{\mathbf{x}}_T^i\}) \leq \mathcal{C}(T, d_T, \{\hat{\mathbf{x}}_T^i\})$. Later in Lemma 2, we showed that $\mathcal{C}(T, d_T, \{\hat{\mathbf{x}}_T^i\}) \leq \mathcal{C}(T, d_T, \{\hat{\mathbf{x}}_T^i\})$. Hence, from (5), the optimum stopping time is the first time $C_t \triangleq \mathcal{C}(t, d_t, \{\hat{\mathbf{x}}_T^i\})$ achieves the target accuracy level α , as shown in (18). Using the optimum decision function d_t at each time t , from (12), we write the optimal cost C_t as in (22). ■

According to Theorem 1, the optimum scheme, at each time t , computes C_t , given by (22), and then compares it to α . When $C_t \leq \alpha$, it stops and makes a decision using (19). Finally, it estimates \mathbf{x} via $\hat{\mathbf{x}}_T^i$, given by (20), if H_i is decided.

Considering the mean-squared error (MSE) as the estimation cost function a pseudo-code for this scheme is given in Algorithm 1. Since the results in Theorem 1 are universal in the sense that they hold for all probability distributions and system models, in Algorithm 1 we provide a general procedure that requires computation of some statistics (cf. lines 4,6,8,10). In specific cases, such statistics may be easily computed. However, in many cases they cannot be written in closed forms, hence intense online computations may be required to estimate them.

Remarks:

- 1) In the sequential detection problem, where only the binary hypothesis testing in (2) is of interest, the classical approach of the well-known sequential probability ratio test (SPRT) [18] fails to provide a feasible optimum solution due to the second observed sequence $\{\mathbf{H}_t\}$. More specifically, observing

Algorithm 1 The procedure for $J(\hat{\mathbf{x}}, \mathbf{x}) = \|\hat{\mathbf{x}} - \mathbf{x}\|^2$

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1: Initialization:  $t \leftarrow 0, \mathcal{C} \leftarrow \infty$ 
2: while  $\mathcal{C} > \alpha$  do
3:    $t \leftarrow t + 1$ 
4:    $p_i = p_i(\{\mathbf{y}_s\}_{s=1}^t | \mathcal{F}_t), i = 0, 1, \dots, I$ 
5:    $L^{ji} = \frac{p_j}{p_i}, i, j = 0, 1, \dots, I$ 
6:    $\mathbf{e}_i = \mathbf{E}_i[\mathbf{x} | \mathcal{G}_t], i = 0, 1, \dots, I$ 
7:    $\hat{\mathbf{x}}^i = \frac{\sum_{j=0}^I b_{ji} L^{ji} \mathbf{e}_i}{\sum_{j=0}^I b_{ji} L^{ji}}, i = 0, 1, \dots, I$ 
8:    $\Delta^{ij} = \text{Tr}(\text{Cov}_i[\mathbf{x} | \mathcal{G}_t]) + \|\hat{\mathbf{x}}^i - \mathbf{e}_i\|^2, i, j = 0, 1, \dots, I$ 
9:    $d = \arg \min_i \sum_{j=0, j \neq i}^I (a_j + b_{ji} \Delta^{ji} - b_{jj} \Delta^{jj}) p_j$ 
10:  Cost:  $\mathcal{C} = \sum_{i=0}^I \mathbf{E}_i[b_{ii} \Delta^{ii} \mathbb{1}_{\{d=i\}}] + \sum_{j=0, j \neq i}^I (a_i + b_{ij} \Delta^{ij}) \mathbb{1}_{\{d=j\}} | \mathcal{F}_t]$ 
11: end while
12: Stop:  $T = t$ 
13: Detection & Estimation: Declare  $d_T = d$  and  $\hat{\mathbf{x}}_T = \hat{\mathbf{x}}^d$ 

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the pair $\{(\mathbf{y}_t, \mathbf{H}_t)\}$ we end up with a two-dimensional optimal stopping problem which is impossible to solve analytically since the thresholds for the running likelihood ratio will depend on the sequence $\{\mathbf{H}_t\}$. On the other hand, for $b_{ij} = 0, i, j = 0, 1$, i.e., in the pure detection problem, the decision function in Theorem 1 boils down to the well-known likelihood ratio test. Using the stopping time given by (18) we obtain a feasible sequential detector. Unlike SPRT, the above sequential detector follows a two-step procedure: it first determines the stopping time using a single threshold, and then decides using another threshold. Whereas, in SPRT, two thresholds are used in a single-step procedure to both stop and decide.

- 2) The optimum scheme given by Theorem 1 is considerably more general and different than the one presented in [9]. Firstly, the estimator here is the optimum estimator under a weighted average of the probability distributions under $\{H_i\}$ since there are unknown parameter vectors under all hypotheses. The weights for the estimator [see (7)] depend on the likelihood ratio L_t^{ji} , hence the detector. That is, the optimum estimator for the general problem introduced in (1)–(5) is coupled with the optimum detector. Whereas, no such coupling exists for the estimator in [9], which is the optimum estimator under H_1 as the unknown parameter appears only under H_1 ($x = 0$ under H_0). Secondly, the optimum detector in (19) is coupled with the estimator through the posterior estimation cost Δ_T^{ji} under the combinations of the true and selected hypotheses. On the other hand, the optimum detector in [9] uses the estimator itself, which is a special case of the detector in (19). Specifically, with $b_{01} = b_{00} = 0$ and $b_{10} = b_{11}$, the optimum estimator is given by $\hat{x}_T = \mathbf{E}_1[x | \mathcal{G}_T]$ when H_1 is decided ($x = 0$ under H_0 , hence $\hat{x}_T = 0$ when H_0 is decided), and accordingly $\Delta_T^{11} = \text{Var}_1[x | \mathcal{G}_T]$, $\Delta_T^{10} = \text{Var}_1[x | \mathcal{G}_T] + \hat{x}_T^2$. Substituting these terms in (19) we obtain the detector in [9, Lemma

2]. Moreover, the scheme presented in Theorem 1 is optimum for a general non-linear model with arbitrary cost function $J(\hat{\mathbf{x}}, \mathbf{x})$, noise distribution, number of parameters, and number of hypotheses; and it covers the optimum scheme in [9] as a special case.

In [9], a monotonicity feature that facilitates the computation of the optimum stopping time is shown after a quite technical proof. Although such a monotonicity feature cannot be shown here due to the generic model we use, the optimum stopping time can still be found through numerical procedures. In the special case studied in [9], the monotonicity also guarantees a finite stopping time if $P(\sum_{t=0}^{\infty} h_t^2 = \infty) = 1$. Here, in general, we need $C_t \rightarrow 0$ almost surely as $t \rightarrow \infty$, which, from (22), implies $\sum_{i=0}^I P_i(d_t \neq i | \mathcal{F}_t) \rightarrow 0$ and $\sum_{i=0}^I E_i[\Delta_t^{ii} | \mathcal{F}_t] \rightarrow 0$ almost surely. In the independent LQG case (analyzed in Section III-A), this boils down to the condition $P(\sum_{t=0}^{\infty} h_{t,n}^2 = \infty) = 1, \forall n$, similar to [9].

- 3) Since our Bayesian cost function, given by (3), is similar to the ones considered in [8], [12] that study fixed-sample-size joint detection and estimation, the structures of the optimum detector and estimator, given by Theorem 1, at the stopping time resemble those derived in [8], [12]. Specifically, the optimum detector presented in [8, Eq. (4.1)] is the counterpart of the specialized version of (19) for binary hypothesis testing; and the optimum estimator given in [12, Eq. (18)] coincides with (21), which is the special case of (20) with quadratic estimation cost. The optimum detector and estimator in Theorem 1 hold for a much more general problem formulation than the ones in [8], [12]. Moreover, solving a sequential joint detection and estimation problem we also provide the optimum stopping time structure given by (18) and (22).
- 4) If the estimation cost is independent of detection decision, i.e., $b_{ij} = b_i, \forall i, j$, then the optimum estimators $\{\hat{\mathbf{x}}_T^i\}$ in (6) are all given by

$$\hat{\mathbf{x}}_T = \arg \min_{\hat{\mathbf{x}}} \bar{E}[J(\hat{\mathbf{x}}, \mathbf{x}) | \mathcal{G}_T], \quad (23)$$

where \bar{E} is the expectation under the distribution

$$\bar{p}_t(\mathbf{x} | \mathcal{G}_t) = \frac{\sum_{j=0}^I b_j p_j(\{\mathbf{y}_s\}_{s=1}^t | \mathcal{F}_t) p_j(\mathbf{x} | \mathcal{G}_t)}{\sum_{j=0}^I b_j p_j(\{\mathbf{y}_s\}_{s=1}^t | \mathcal{F}_t)}. \quad (24)$$

In particular, the MMSE estimators in (10) become

$$\hat{\mathbf{x}}_T = \frac{\sum_{j=0}^I b_j p_j(\{\mathbf{y}_t\}_{t=1}^T | \mathcal{F}_T) E_j[\mathbf{x} | \mathcal{G}_T]}{\sum_{j=0}^I b_j p_j(\{\mathbf{y}_t\}_{t=1}^T | \mathcal{F}_T)}, \quad (25)$$

regardless of the detection decision. Note that the optimum estimator is still coupled with the detector through the likelihood $p_j(\{\mathbf{y}_t\}_{t=1}^T | \mathcal{F}_T)$. On the other hand, the optimum detector becomes the maximum likelihood test (likelihood ratio test for the binary case), hence independent of the

estimator, since $\Delta_T^{i0} = \Delta_T^{i1} = \mathbb{E}_i [J(\hat{\mathbf{x}}_T, \mathbf{x}) | \mathcal{G}_T] = \Delta_T^i$, as well as $b_{ij} = b_i$.

C. Separated Detection and Estimation Costs

In the combined cost function, given by (3), if we penalize the wrong decisions only with the detection costs, i.e., $b_{ij} = 0, i \neq j$, we get the following simplified alternative cost function

$$\mathcal{C}(T, d_T, \{\hat{\mathbf{x}}_T^i\}) = \sum_{i=0}^I a_i \mathbb{P}_i(d_T \neq i | \mathcal{F}_T) + b_i \mathbb{E}_i [J(\hat{\mathbf{x}}_T^i, \mathbf{x}) \mathbb{1}_{\{d_T=i\}} | \mathcal{F}_T]. \quad (26)$$

In this alternative form, detection and estimation costs are used to penalize separate cases. Specifically, under H_i , the wrong decision case is penalized with the constant detection cost a_i , and the correct decision case is penalized with the estimation cost $b_i \mathbb{E}_i [J(\hat{\mathbf{x}}_T^i, \mathbf{x}) | \mathcal{F}_T]$. Since a_i is the only cost to penalize the wrong decision case, it is typically assigned a larger number here than in (3).

The optimum scheme is obtained by substituting $b_{ij} = 0, i \neq j$, in Theorem 1.

Corollary 1. *Considering the combined cost function with separated detection and estimation costs, given by (26), the optimum sequential joint detector and estimator $(\mathbb{T}, d_{\mathbb{T}}, \hat{\mathbf{x}}_{\mathbb{T}})$ for the problem in (5) is given by*

$$\mathbb{T} = \min\{t \in \mathbb{N} : \mathcal{C}_t \leq \alpha\} \quad (27)$$

$$d_{\mathbb{T}} = \arg \max_i (a_i - b_i \Delta_{\mathbb{T}}^i) p_i(\{\mathbf{y}_t\}_{t=1}^{\mathbb{T}} | \mathcal{F}_{\mathbb{T}}) \quad (28)$$

$$\hat{\mathbf{x}}_{\mathbb{T}} = \arg \min_{\hat{\mathbf{x}}} \mathbb{E}_{d_{\mathbb{T}}} [J(\hat{\mathbf{x}}, \mathbf{x}) | \mathcal{G}_{\mathbb{T}}] \quad (29)$$

$$\left(\text{e.g., } \hat{\mathbf{x}}_{\mathbb{T}} = \mathbb{E}_{d_{\mathbb{T}}} [\mathbf{x} | \mathcal{G}_{\mathbb{T}}] \text{ for } J(\hat{\mathbf{x}}, \mathbf{x}) = \|\hat{\mathbf{x}} - \mathbf{x}\|^2 \right), \quad (30)$$

where

$$\mathcal{C}_t = \sum_{i=0}^I \mathbb{E}_i [a_i \mathbb{1}_{\{d_t \neq i\}} + b_i \Delta_t^i \mathbb{1}_{\{d_t=i\}} | \mathcal{F}_t], \quad (31)$$

is the optimal cost at time t .

The optimum stopping time, given in (27), has the same structure as in Theorem 1, with a simplified optimal cost, given in (31).

Since here we are not interested in minimizing the estimation costs in case of wrong decisions, when we decide H_i , we use the optimum estimator under H_i [cf. (29)]. Recall that in Theorem 1, the optimum estimator is a mixture of the optimum estimators under all hypotheses. Consequently, the posterior expected estimation cost in the correct decision case achieves the minimum, i.e.,

$$\Delta_{\mathbb{T}}^i = \min_{\hat{\mathbf{x}}} \mathbb{E}_i [J(\hat{\mathbf{x}}, \mathbf{x}) | \mathcal{G}_{\mathbb{T}}]. \quad (32)$$

For the MSE criterion, with $J(\hat{\mathbf{x}}, \mathbf{x}) = \|\hat{\mathbf{x}} - \mathbf{x}\|^2$,

$$\Delta_{\mathsf{T}}^i = \text{Tr}(\text{Cov}_i[\mathbf{x}|\mathcal{G}_{\mathsf{T}}]) = \text{MMSE}_{\mathsf{T},i}. \quad (33)$$

On the other hand, in the wrong decision case, which is not of interest here, the posterior estimation cost Δ_{T}^{ij} , $i \neq j$, is higher than that in Theorem 1.

The optimum detector in (28) is biased towards the hypothesis with better estimation performance. For instance, when the minimum posterior estimation cost Δ_{T}^i (e.g., MMSE) under H_i is small, it is easier to decide in favor of H_i . Conversely, large Δ_{T}^i makes it difficult to choose H_i . Considering the MSE estimation cost we can call it *ML & MMSE detector* since it uses the maximum likelihood (ML) criterion, as in the likelihood ratio test, together with the MMSE criterion.

III. LINEAR QUADRATIC GAUSSIAN (LQG) MODEL

In this section, we consider the commonly used linear quadratic Gaussian (LQG) model under binary hypothesis testing, as a special case. In particular, we have the quadratic (i.e., MSE) estimation cost

$$J(\hat{\mathbf{x}}, \mathbf{x}) = \|\hat{\mathbf{x}} - \mathbf{x}\|^2, \quad (34)$$

and the linear system model

$$\mathbf{y}_t = \mathbf{H}_t \mathbf{x} + \mathbf{w}_t, \quad (35)$$

where $\mathbf{H}_t \in \mathbb{R}^{M \times N}$, \mathbf{w}_t is the white Gaussian noise with covariance $\sigma^2 \mathbf{I}$, and \mathbf{x} is Gaussian under both hypotheses, i.e.,

$$H_0 : \mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0), \quad (36)$$

$$H_1 : \mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1).$$

We next derive the closed-form expressions for the sufficient statistics for the optimum scheme presented in Theorem 1. Using (37)–(40), the optimum stopping time, detector, and estimator can be computed as in (18), (19), and (21), respectively.

Proposition 1. *Considering the LQG model in (34)–(36), the sufficient statistics for the optimum sequential joint detector and estimator, presented in Theorem 1, namely the conditional mean $\mathbb{E}_i[\mathbf{x}|\mathcal{G}_{\mathsf{T}}]$, the posterior estimation cost $\Delta_{\mathsf{T}}^{ij} = \mathbb{E}_i[\|\mathbf{x} - \hat{\mathbf{x}}_{\mathsf{T}}^j\|^2|\mathcal{G}_{\mathsf{T}}]$ for deciding H_j under H_i , and the likelihood ratio $L_{\mathsf{T}} = \frac{p_1(\{\mathbf{y}_t\}_{t=1}^{\mathsf{T}}|\mathcal{F}_{\mathsf{T}})}{p_0(\{\mathbf{y}_t\}_{t=1}^{\mathsf{T}}|\mathcal{F}_{\mathsf{T}})}$ are written as*

$$\mathbb{E}_i[\mathbf{x}|\mathcal{G}_{\mathsf{T}}] = \left(\frac{\mathbf{U}_{\mathsf{T}}}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1} \right)^{-1} \left(\frac{\mathbf{v}_{\mathsf{T}}}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\mu}_i \right), \quad (37)$$

$$\Delta_{\mathsf{T}}^{ij} = \text{Tr} \left(\left(\frac{\mathbf{U}_{\mathsf{T}}}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1} \right)^{-1} \right) + \delta_{\mathsf{T}}^{ij} \|\mathbb{E}_0[\mathbf{x}|\mathcal{G}_{\mathsf{T}}] - \mathbb{E}_1[\mathbf{x}|\mathcal{G}_{\mathsf{T}}]\|^2, \quad (38)$$

$$L_{\mathbf{T}} = \sqrt{\frac{|\Sigma_0| \left| \frac{\mathbf{U}_{\mathbf{T}}}{\sigma^2} + \Sigma_0^{-1} \right|}{|\Sigma_1| \left| \frac{\mathbf{U}_{\mathbf{T}}}{\sigma^2} + \Sigma_1^{-1} \right|}} \exp \left[\frac{1}{2} \left(\left\| \frac{\mathbf{v}_{\mathbf{T}}}{\sigma^2} + \Sigma_1^{-1} \boldsymbol{\mu}_1 \right\|^2 \left(\frac{\mathbf{U}_{\mathbf{T}}}{\sigma^2} + \Sigma_1^{-1} \right)^{-1} - \left\| \frac{\mathbf{v}_{\mathbf{T}}}{\sigma^2} + \Sigma_0^{-1} \boldsymbol{\mu}_0 \right\|^2 \left(\frac{\mathbf{U}_{\mathbf{T}}}{\sigma^2} + \Sigma_0^{-1} \right)^{-1} + \|\boldsymbol{\mu}_0\|_{\Sigma_0^{-1}}^2 - \|\boldsymbol{\mu}_1\|_{\Sigma_1^{-1}}^2 \right) \right], \quad (39)$$

where $\|\mathbf{x}\|_{\Sigma}^2 \triangleq \mathbf{x}'\Sigma\mathbf{x}$,

$$\mathbf{U}_{\mathbf{T}} \triangleq \sum_{t=1}^{\mathbf{T}} \mathbf{H}_t' \mathbf{H}_t, \quad \mathbf{v}_{\mathbf{T}} \triangleq \sum_{t=1}^{\mathbf{T}} \mathbf{H}_t' \mathbf{y}_t, \quad (40)$$

$$\delta_{\mathbf{T}}^{0j} = \left(\frac{b_{1j} L_{\mathbf{T}}}{b_{0j} + b_{1j} L_{\mathbf{T}}} \right)^2 \quad \text{and} \quad \delta_{\mathbf{T}}^{1j} = \left(\frac{b_{0j}}{b_{0j} + b_{1j} L_{\mathbf{T}}} \right)^2. \quad (41)$$

Proof: See Appendix C. ■

Note that the sufficient statistics in (37)–(39) are functions of $\mathbf{U}_{\mathbf{T}}$ and $\mathbf{v}_{\mathbf{T}}$ only, which are given in (40). As a result, from (22), given \mathcal{F}_t , the expectation in the optimal cost \mathcal{C}_t is conditional on \mathbf{U}_t as \mathbf{U}_t is \mathcal{F}_t -measurable, and hence the expectation is taken over \mathbf{v}_t . That is, \mathcal{C}_t and the optimum stopping time \mathbf{T} , given by (22) and (18), respectively, are functions of \mathbf{U}_t only, which is in fact the Fisher information matrix scaled by σ^2 .

Using (35) and (40) we can write

$$\mathbf{v}_t = \mathbf{U}_t \mathbf{x} + \sum_{s=1}^t \mathbf{H}_s' \mathbf{w}_s, \quad (42)$$

which is distributed as $\mathcal{N}(\mathbf{U}_t \boldsymbol{\mu}_i, \mathbf{U}_t \Sigma_i \mathbf{U}_t + \sigma^2 \mathbf{U}_t)$ under H_i . At each time t , for the corresponding \mathbf{U}_t , we can estimate the optimal cost \mathcal{C}_t through Monte Carlo simulations, and stop if $\mathcal{C}_t \leq \alpha$ according to (18). Specifically, given \mathbf{U}_t we generate realizations of \mathbf{v}_t , compute the expression inside the expectation in (22) using (37)–(39), and average them. Alternatively, $\mathcal{C}(\mathbf{U})$ can be computed in the same way through offline Monte Carlo simulations on a grid of \mathbf{U} . Then, at each time t , checking the $\mathcal{C}(\mathbf{U}^*)$ value for the average \mathbf{U}^* of $2^{\frac{N^2+N}{2}}$ neighboring points to \mathbf{U}_t (or simply the closest grid point \mathbf{U}^* to \mathbf{U}_t) we can decide to stop if $\mathcal{C}(\mathbf{U}^*) \leq \alpha$ or to continue if $\mathcal{C}(\mathbf{U}^*) > \alpha$. Although $\mathbf{U}_t = \sum_{s=1}^t \mathbf{H}_s' \mathbf{H}_s$ has N^2 entries, due to symmetry the grid for offline simulations is $\frac{N^2+N}{2}$ -dimensional.

A. Independent LQG Model

Here, we further assume in (35) that the entries of \mathbf{x} are independent [i.e., Σ_0 and Σ_1 are diagonal in (36)], and \mathbf{H}_t is diagonal. Note that in this case $M = N$, and the entries of \mathbf{y}_t are independent. This may be the case in a distributed system (e.g., wireless sensor network) in which each node (e.g., sensor) takes noisy measurements of a local parameter, and there is a global event whose occurrence changes the

probability distributions of local parameters. In such a setup, nodes collaborate through a fusion center to jointly detect the global event and estimate the local parameters. To find the optimal scheme we assume that all the observations collected at nodes are available to the fusion center.

Proposition 2. *Considering the independent LQG model with diagonal \mathbf{H}_t and Σ_i in (35) and (36), respectively, the necessary and sufficient statistics for the optimum scheme in Theorem 1 are written as*

$$\mathbf{E}_i[\mathbf{x}|\mathcal{G}_T] = [\bar{x}_1, \dots, \bar{x}_N]', \quad \bar{x}_n = \frac{\frac{v_{T,n}}{\sigma^2} + \frac{\mu_{i,n}}{\rho_{i,n}^2}}{\frac{u_{T,n}}{\sigma^2} + \frac{1}{\rho_{i,n}^2}}, \quad (43)$$

$$\Delta_T^{ij} = \sum_{n=1}^N \frac{1}{\frac{u_{T,n}}{\sigma^2} + \frac{1}{\rho_{i,n}^2}} + \delta_T^{ij} \|\mathbf{E}_0[\mathbf{x}|\mathcal{G}_T] - \mathbf{E}_1[\mathbf{x}|\mathcal{G}_T]\|^2, \quad (44)$$

$$L_T = \prod_{n=1}^N \frac{\rho_{0,n}}{\rho_{1,n}} \sqrt{\frac{\frac{u_{T,n}}{\sigma^2} + \frac{1}{\rho_{0,n}^2}}{\frac{u_{T,n}}{\sigma^2} + \frac{1}{\rho_{1,n}^2}}} \exp \left[\frac{1}{2} \left(\frac{\left(\frac{v_{T,n}}{\sigma^2} + \frac{\mu_{1,n}}{\rho_{1,n}^2} \right)^2}{\frac{u_{T,n}}{\sigma^2} + \frac{1}{\rho_{1,n}^2}} - \frac{\left(\frac{v_{T,n}}{\sigma^2} + \frac{\mu_{0,n}}{\rho_{0,n}^2} \right)^2}{\frac{u_{T,n}}{\sigma^2} + \frac{1}{\rho_{0,n}^2}} + \frac{\mu_{0,n}^2}{\rho_{0,n}^2} - \frac{\mu_{1,n}^2}{\rho_{1,n}^2} \right) \right], \quad (45)$$

where δ_T^{ij} is given by (41), the subscript n denotes the n -th entry of the corresponding vector, $\rho_{i,n}^2$ and $h_{t,n}$ are the n -th diagonal entries of Σ_i and \mathbf{H}_t , respectively,

$$u_{T,n} = \sum_{t=1}^T h_{t,n}^2 \quad \text{and} \quad v_{T,n} = \sum_{t=1}^T h_{t,n} y_{t,n}. \quad (46)$$

Proof: See Appendix D. ■

In this case, $\mathbf{E}_i[\mathbf{x}|\mathcal{G}_t]$, Δ_t^{ij} , and L_t are functions of $\{u_{t,n}, v_{t,n}\}_{n=1}^N$ only, hence the optimal cost \mathcal{C}_t and the optimum stopping time T , given in Theorem 1, are functions of $\{u_{t,n}\}_{n=1}^N$ only. At each time t , given $\{u_{t,n}\}_{n=1}^N$, we can estimate \mathcal{C}_t through Monte Carlo simulations using

$$v_{t,n} \sim \mathcal{N}(\mu_{i,n} u_{t,n}, \rho_{i,n}^2 u_{t,n}^2 + \sigma^2 u_{t,n}), \quad (47)$$

and (22), (43)–(45); and stop when the estimated $\mathcal{C}_t \leq \alpha$. Alternatively, $\mathcal{C}(\{u_{t,n}\})$ can be computed in the same way through offline Monte Carlo simulations on a grid of $\{u_{t,n}\}_{n=1}^N$, as discussed in the general LQG case. Note that the grid here is N -dimensional, which is much smaller than the $\frac{N^2+N}{2}$ -dimensional grid under the general LQG model. Consequently, the alternative scheme that performs offline simulations is more viable here.

Remark (Computational complexity for \mathcal{C}_t):

At each time t in the online computation of the optimal cost \mathcal{C}_t (or on each grid point in the offline computation), the complexity with respect to the number of hypothesis I scales (i) quadratically in the general case given by (22), (ii) linearly in the special case of (31); and the complexity with respect to

the number of unknown parameters N scales (i) quadratically for the LQG model given by Proposition 1, (ii) linearly for the independent LQG model given by Proposition 2. Since the computation of \mathcal{C}_t is the most expensive part of the proposed algorithm given in Algorithm 1, the computational complexity given in Table I for the four possible models holds for the whole algorithm, as well as \mathcal{C}_t . In all cases, the computational complexity linearly scales with the number of Monte Carlo iterations.

TABLE I

COMPUTATIONAL COMPLEXITY FOR THE PROPOSED ALGORITHM (SEE ALGORITHM 1) FOR THE FOUR COMBINATIONS OF MODELS DISCUSSED IN THEOREM 1/COROLLARY 1 AND PROPOSITION 1/PROPOSITION 2

	LQG (37)-(41)	Independent LQG (43)-(46)
General cost (3)	$O(I^2 N^2)$	$O(I^2 N)$
Separated cost (26)	$O(IN^2)$	$O(IN)$

B. Numerical Results

In this subsection, we compare the proposed sequential joint detection and estimation (SJDE) scheme with the conventional method, which invokes the sequential detector to decide between the two hypotheses and then computes the corresponding MMSE estimate. The comparison is based on the LQG model presented in (34)–(36). In particular, for the conventional method, the famous sequential probability ratio test (SPRT) is used, followed by an MMSE estimator. SPRT computes the log-likelihood ratio, i.e., $\log L_t$, given by (39), at each sampling instant and examines whether it falls in the prescribed interval, denoted as $[-B, A]$. The stopping time and decision rule of SPRT are defined as

$$\tau_{\text{SPRT}} \triangleq \min \{t \in \mathbb{N} : \log L_t \in [-B, A]\}, \quad (48)$$

$$\text{and } d_{\tau_{\text{SPRT}}} = \begin{cases} 1 & \text{if } \log L_{\tau_{\text{SPRT}}} \geq A, \\ 0 & \text{if } \log L_{\tau_{\text{SPRT}}} \leq -B, \end{cases} \quad (49)$$

where A and B are selected such that the constraints on the false alarm and misdetection probabilities are satisfied. Upon the decision $d_{\tau_{\text{SPRT}}}$ is made, the corresponding MMSE estimator follows. In fact, SPRT is not feasible here since L_t , and thus A and B depend on $\{\mathbf{H}_t\}$, which is random. However, for comparison purposes we run the conventional scheme by selecting A and B regardless of any constraints. The average stopping time and accuracy level are recorded after many runs.

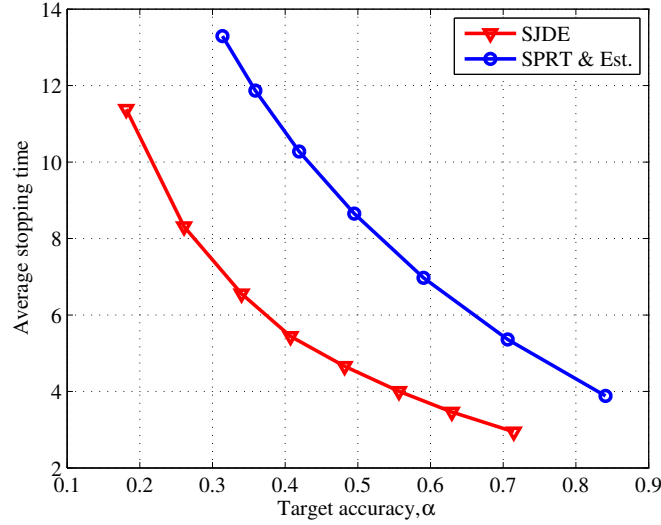


Fig. 1. Average stopping time vs. target accuracy level for SJDE and the combination of SPRT detector & MMSE estimator.

We consider the LQG model (35) with $\mathbf{H}_t^{1 \times 3} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$, $w_t \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$, and the following hypotheses,

$$\begin{aligned} H_0 : \mathbf{x}^{3 \times 1} &\sim \mathcal{N}(\mathbf{1}, 0.5\mathbf{I}), \\ H_1 : \mathbf{x}^{3 \times 1} &\sim \mathcal{N}(-\mathbf{1}, 0.5\mathbf{I}), \end{aligned} \quad (50)$$

where $\mathbf{1}$ is the vector of ones and \mathbf{I} is the identity matrix. \mathbf{H}_t and w_t are independent and identically distributed (iid) over time. The parameters of the cost function are set as follows: $a_0 = a_1 = 0.5$, $b_{00} = b_{11} = 0.5$, $b_{10} = b_{01} = 0$. Fig. 1 illustrates the performance of SJDE and the conventional scheme (SPRT & Est.) in terms of the average stopping time against the target accuracy level α . Note that small α implies high accuracy on the detection and estimation performance, thus requires larger stopping time. It is seen that SJDE significantly outperforms the conventional scheme. That is, SJDE exhibits a much smaller average stopping time, while achieving the same target accuracy level α .

IV. DYNAMIC SPECTRUM ACCESS IN COGNITIVE RADIO NETWORKS

A. Background

Dynamic spectrum access is a fundamental problem in cognitive radio, in which secondary users (SUs) are allowed to utilize a wireless spectrum band (i.e., communication channel) that is licensed to primary users (PUs) without affecting the PU quality of service (QoS) [32]. Spectrum sensing plays a key role in maximizing the SU throughput, and at the same time protecting the PU QoS. In spectrum sensing, if no PU communication is detected, then SU can opportunistically utilize the band [33], [34]. Otherwise, it has to meet some strict interference constraints. Nevertheless, it can still use the band in an underlay

fashion with a transmit power that does not violate the maximum allowable interference level [35], [36]. Methods for combining the underlay and opportunistic access approaches have also been proposed, e.g., [1], [37], [38]. In such combined methods, the SU senses the spectrum band, as in opportunistic access, and controls its transmit power using the sensing result, which allows SU to coexist with PU, as in underlay.

The interference at the PU receiver is a result of the SU transmit power, and also the power gain of the channel between the SU transmitter and PU receiver. Hence, SU needs to estimate the channel coefficient to keep its transmit power within allowable limits. As a result, channel estimation, in addition to PU detection, is an integral part of an effective dynamic spectrum access scheme in cognitive radio. In spectrum access methods it is customary to assume perfect channel state information (CSI) at the SU, e.g., [35]–[37]. It is also crucial to minimize the sensing time for maximizing the SU throughput. Specifically, decreasing the sensing period, that is used to determine the transmit power, saves time for data communication, increasing the SU throughput. Consequently, dynamic spectrum access in cognitive radio is intrinsically a sequential joint detection and estimation problem. Recently, in [1], the joint problem of PU detection and channel estimation for SU power control has been addressed using a sequential two-step procedure. In the first step, sequential joint spectrum sensing and channel estimation is performed; and in the second stage, the SU transmit power is determined based on the results of first stage. Here, omitting the second stage, we derive the optimum scheme for the first stage in an alternative way under the general theory presented in the previous sections.

B. Problem Formulation

We consider a cognitive radio network consisting of K SUs, and a pair of PUs. In PU communication, a preamble takes place before data communication for synchronization and channel estimation purposes. In particular, during the preamble both PUs transmit random pilot symbols simultaneously through full duplexing. Pilot signals are often used in channel estimation, e.g., [39], and also in spectrum sensing, e.g., [40]. We assume each SU observes such pilot symbols (e.g., it knows the seed of the random number generator) so that it can estimate the channels between itself and PUs. Moreover, SUs cooperate to detect the PU communication, through a fusion center (FC), which can be one of the SUs. To find the optimal scheme we assume a centralized setup where all the observations collected at SUs are available to the FC. In practice, under stringent energy and bandwidth constraints SUs can effectively report their necessary and sufficient statistics to the FC using a non-uniform sampling technique called level-triggered sampling, as proposed in [1].

When the channel is idle (i.e., no PU communication), there is no interference constraint, and as a result SUs do not need to estimate the interference channels to determine the transmit power, which is simply the full power P_{\max} . On the other hand, in the presence of PU communication, to satisfy the peak interference power constraints I_1 and I_2 of PU 1 and PU 2, respectively, SU k should transmit with power

$$P_k = \min \left\{ P_{\max}, \frac{I_1}{x_{1k}^2}, \frac{I_2}{x_{2k}^2} \right\}, \quad (51)$$

where x_{jk} is the channel coefficient between PU j and SU k . Hence, firstly the presence/absence of PU communication is detected. If no PU communication is detected, then a designated SU transmits data with P_{\max} . Otherwise, the channels between PUs and SUs are estimated to determine transmission powers, and then the SU with the highest transmission power starts data communication.

We can model this sequential joint detection and estimation problem using the linear model in (35), where the vector

$$\mathbf{x} = [x_{11}, \dots, x_{1K}, x_{21}, \dots, x_{2K}]' \quad (52)$$

holds the interference channel coefficients between PUs ($j = 1, 2$) and SUs ($k = 1, \dots, K$); the diagonal matrix

$$\mathbf{H}_t = \text{diag}(h_{t,1}, \dots, h_{t,1}, h_{t,2}, \dots, h_{t,2}) \in \mathbb{R}^{2K \times 2K} \quad (53)$$

holds the PU pilot signals; and

$$\mathbf{y}_t = [y_{t,11}, \dots, y_{t,2K}]' \quad (54)$$

$$\mathbf{w}_t = [w_{t,11}, \dots, w_{t,2K}]' \quad (55)$$

are the observation and Gaussian noise vectors at time t , respectively. Then, we have the following binary hypothesis testing problem

$$H_0 : \mathbf{x} = 0, \quad (56)$$

$$H_1 : \mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}),$$

where $\boldsymbol{\mu} = [\mu_{11}, \dots, \mu_{2K}]'$, $\boldsymbol{\Sigma} = \text{diag}(\rho_{11}^2, \dots, \rho_{2K}^2)$ with μ_{jk} and ρ_{jk}^2 being the mean and variance of the channel coefficient x_{jk} , respectively.

Since channel estimation is meaningful only under H_1 , we do not assign estimation cost to H_0 , and perform estimation only when H_1 is decided. In other words, we use the cost function

$$\begin{aligned} \mathcal{C}(T, d_T, \hat{\mathbf{x}}_T) = & a_0 P_0(d_T = 1 | \mathcal{F}_T) + a_1 P_1(d_T = 0 | \mathcal{F}_T) \\ & + b_1 E_1 [\|\hat{\mathbf{x}}_T - \mathbf{x}\|^2 \mathbb{1}_{\{d_T=1\}} + \|\mathbf{x}\|^2 \mathbb{1}_{\{d_T=0\}} | \mathcal{F}_T], \end{aligned} \quad (57)$$

which is a special case of (3). When H_0 is decided, it is like we set $\hat{\mathbf{x}}_T = 0$. Similar to (5), we want to solve the following problem

$$\min_{T, d_T, \hat{\mathbf{x}}_T} T \quad \text{s.t.} \quad \mathcal{C}(T, d_T, \hat{\mathbf{x}}_T) \leq \alpha, \quad (58)$$

for which the optimum solution follows from Theorem 1 and Proposition 2.

C. Optimum Solution

Corollary 2. *The optimum scheme for the sequential joint spectrum sensing and channel estimation problem in (58) is given by*

$$T = \min\{t \in \mathbb{N} : \mathcal{C}_t \leq \alpha\} \quad (59)$$

$$d_T = \begin{cases} 1 & \text{if } L_T \geq \frac{a_0}{a_1 + b_1 \|\hat{\mathbf{x}}_T\|^2} \\ 0 & \text{otherwise} \end{cases} \quad (60)$$

$$\hat{\mathbf{x}}_T = [\bar{x}_{11}, \dots, \bar{x}_{2K}]', \quad \text{and} \quad \bar{x}_{jk} = \frac{\frac{v_{T,jk}}{\sigma^2} + \frac{\mu_{jk}}{\rho_{jk}^2}}{\frac{u_{T,j}}{\sigma^2} + \frac{1}{\rho_{jk}^2}}, \quad (61)$$

where $u_{T,j} = \sum_{t=1}^T h_{t,j}^2$, $v_{T,jk} = \sum_{t=1}^T h_{t,j} y_{t,jk}$,

$$\mathcal{C}_t = \mathbb{E}_0 \left[\left\{ a_0 - (a_1 + b_1 \|\hat{\mathbf{x}}_t\|^2) L_t \right\}^- + b_1 L_t \|\hat{\mathbf{x}}_t\|^2 | \mathcal{F}_t \right] + a_1 + b_1 \sum_{j=1}^2 \sum_{k=1}^K \frac{1}{\frac{u_{t,j}}{\sigma^2} + \frac{1}{\rho_{jk}^2}} \quad (62)$$

is the optimal cost at time t , $A^- = \min(0, A)$; and

$$L_t = \frac{p_1(\{\mathbf{y}_s\}_{s=1}^t | \mathcal{F}_t)}{p_0(\{\mathbf{y}_s\}_{s=1}^t | \mathcal{F}_t)} = \prod_{j=1}^2 \prod_{k=1}^K \frac{\exp \left[\frac{1}{2} \left(\frac{\left(\frac{v_{t,jk}}{\sigma^2} + \frac{\mu_{jk}}{\rho_{jk}^2} \right)^2}{\frac{u_{t,j}}{\sigma^2} + \frac{1}{\rho_{jk}^2}} - \frac{\mu_{jk}^2}{\rho_{jk}^2} \right) \right]}{\rho_{jk} \sqrt{\frac{u_{t,j}}{\sigma^2} + \frac{1}{\rho_{jk}^2}}} \quad (63)$$

is the likelihood ratio at time t .

Proof: See Appendix E. ■

At each time t the optimal cost \mathcal{C}_t , given by (62), can be estimated through Monte Carlo simulations by generating the realizations of $v_{t,jk}$, independently for each pair (j, k) , according to $\mathcal{N}(0, \sigma^2 u_{t,j})$ and $\mathcal{N}(\mu_{jk} u_{t,j}, \rho_{jk}^2 u_{t,j}^2 + \sigma^2 u_{t,j})$ under H_0 and H_1 , respectively. Alternatively, since \mathcal{C}_t is a function of $u_{t,1}$ and $u_{t,2}$ only, we can effectively estimate $\mathcal{C}(u_1, u_2)$ through offline Monte Carlo simulations over the 2-dimensional grid. Note that the number of grid dimensions here is much less than N and $\frac{N^2+N}{2}$ for the independent and general LQG models in Section III, respectively.

The optimum detector, given in (60), uses the side information provided by the estimator itself. Specifically, the farther away the estimates are from zero, i.e., $\|\hat{\mathbf{x}}_T\|^2 \gg 0$, the easier it is to decide

for H_1 ; and the reverse is true for H_0 . The optimum estimator, given by (61), is the MMSE estimator under H_1 as channel estimation is meaningful only when PU communication takes place.

Remark: In [1], following the technical proof of [9] the optimum solution is presented for a similar sequential joint detection and estimation problem with complex channels. Here, under a general framework, we derive the optimum scheme following an alternative approach. Particularly, we show that, without the monotonicity property for the optimal cost, the optimum stopping time can be efficiently computed through (offline/online) Monte Carlo simulations. Furthermore, we here also show how this dynamic spectrum access method fits to the systematic theory of sequential joint detection and estimation, developed in the previous sections.

V. STATE ESTIMATION IN SMART GRID WITH TOPOLOGICAL UNCERTAINTY

A. Background and Problem Formulation

State estimation is a vital task in real-time monitoring of smart grid [42]. In the widely used linear model

$$\mathbf{y}_t = \mathbf{H}\mathbf{x} + \mathbf{w}_t, \quad (64)$$

the state vector $\mathbf{x} = [\theta_1, \dots, \theta_N]'$ holds the bus voltage phase angles; the measurement matrix $\mathbf{H} \in \mathbb{R}^{M \times N}$ represents the network topology; $\mathbf{y}_t \in \mathbb{R}^M$ holds the power flow and injection measurements; and $\mathbf{w}_t \in \mathbb{R}^M$ is the white Gaussian measurement noise vector. We assume a pseudo-static state estimation problem, i.e., \mathbf{x} does not change during the estimation period. For the above linear model to be valid it is assumed that the differences between phase angles are small. Hence, we can model θ_n , $n = 1, \dots, N$ using a Gaussian prior with a small variance, as in [2], [43].

The measurement matrix \mathbf{H} is also estimated periodically using the status data from switching devices in the power grid, and assumed to remain unchanged until the next estimation instance. However, in practice, such status data is also noisy, like the power flow measurements in (64), and thus the estimate of \mathbf{H} may include some error. Since the elements of \mathbf{H} take the values $\{-1, 0, 1\}$, there is a finite number of possible errors. Another source of topological uncertainty is the power outage, in which protective devices automatically isolate the faulty area from the rest of the grid. Specifically, an outage changes the grid topology, i.e., \mathbf{H} , and also the prior on \mathbf{x} . We model the topological uncertainty using multiple hypotheses, as in [2], [44]–[46]. In (64), under hypothesis i we have

$$\mathbf{H}_i : \quad \mathbf{H} = \mathbf{H}_i, \quad \text{and} \quad \mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i), \quad i = 0, 1, \dots, I, \quad (65)$$

where H_0 corresponds to the normal-operation (i.e., no estimation error or outage) case.

For simplicity, following the formulation in Section II-C, we here penalize the wrong decisions only with the detection costs, i.e., $b_{ji} = 0$, $i \neq j$, and $b_{ii} = b_i > 0$. Hence, we use the cost function

$$\mathcal{C}(T, d_T, \{\hat{\mathbf{x}}_T^i\}) = \sum_{i=0}^I a_i \mathbf{P}_i(d_T \neq i | \mathcal{F}_T) + b_i \mathbf{E}_i [J(\hat{\mathbf{x}}_T^i, \mathbf{x}) \mathbb{1}_{\{d_T=i\}} | \mathcal{F}_T], \quad (66)$$

as in (26). Here we do not need the conditioning on \mathcal{F}_t as the measurement matrices $\{\mathbf{H}_j\}$ are deterministic and known. As a result the optimum stopping time T is deterministic and can be computed offline. We seek the solution to the following optimization problem,

$$\min_{T, d_T, \{\hat{\mathbf{x}}_T^j\}} T \quad \text{s.t.} \quad \mathcal{C}(T, d_T, \{\hat{\mathbf{x}}_T^j\}) \leq \alpha. \quad (67)$$

B. Optimum Solution

Corollary 3. *The optimum scheme for the sequential joint detection and estimation problem in (67) is given by*

$$T = \min\{t \in \mathbb{N} : C_t \leq \alpha\}, \quad (68)$$

$$d_T = \arg \max_i (a_i - b_i \Delta_T^i) p_i \left(\{\mathbf{y}_t\}_{t=1}^T \right), \quad (69)$$

$$\hat{\mathbf{x}}_T = \left(\frac{\mathbf{U}_{T, d_T}}{\sigma^2} + \boldsymbol{\Sigma}_{d_T}^{-1} \right)^{-1} \left(\frac{\mathbf{v}_{T, d_T}}{\sigma^2} + \boldsymbol{\Sigma}_{d_T}^{-1} \boldsymbol{\mu}_{d_T} \right), \quad (70)$$

where $\mathbf{U}_{t,i} = t \mathbf{H}_i' \mathbf{H}_i$ and $\mathbf{v}_{t,i} = \mathbf{H}_i' \sum_{s=1}^t \mathbf{y}_s$,

$$C_t = \sum_{i=0}^I (a_i - b_i \Delta_t^i) \mathbf{P}_i(d_t \neq i) + b_i \Delta_t^i \quad (71)$$

is the optimal cost at time t ;

$$\Delta_T^i = \text{Tr} \left(\left(\frac{\mathbf{U}_{T,i}}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1} \right)^{-1} \right) \quad (72)$$

is the MMSE under H_i at time T ;

$$p_i \left(\{\mathbf{y}_t\}_{t=1}^T \right) = \frac{\exp \left[-\frac{1}{2} \left(\sum_{t=1}^T \frac{\|\mathbf{y}_t\|^2}{\sigma^2} + \|\boldsymbol{\mu}_i\|_{\boldsymbol{\Sigma}_i^{-1}}^2 - \left\| \frac{\mathbf{v}_{T,i}}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\mu}_i \right\|^2 \left(\frac{\mathbf{U}_{T,i}}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1} \right)^{-1} \right) \right]}{(2\pi)^{mT/2} \sigma^{mT} |\boldsymbol{\Sigma}_i|^{1/2} \left| \frac{\mathbf{U}_{T,i}}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1} \right|^{1/2}} \quad (73)$$

is the likelihood under H_i at time T .

Proof: See Appendix F. ■

The optimal cost C_t can be numerically computed offline for each t by estimating the sufficient statistics $\{\mathbf{P}_i(d_t \neq i)\}$ through Monte Carlo simulations. Specifically, under H_j , we can independently generate the samples of \mathbf{x} and $\{\mathbf{w}_1, \dots, \mathbf{w}_t\}$, and compute d_t as in (69). Then, the ratio of the number of instances

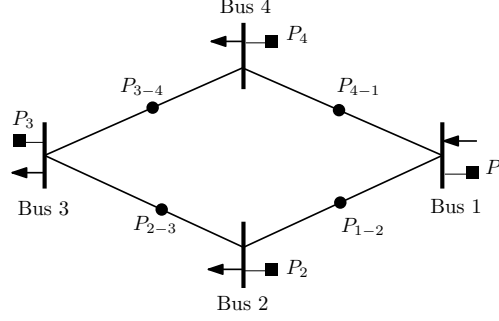


Fig. 2. Illustration for the IEEE-4 bus system with the power injection (square) and power flow (circle) measurements.

in which $d_t \neq i$ to the total number of instances gives an estimate of the probability $P_i(d_t \neq i)$. Once the sequence $\{\mathcal{C}_t\}$ is obtained, the optimum detection and estimation time is found offline using (68).

As in Section II-C, the optimum detector in (69) is biased towards the hypothesis with best estimation performance (i.e., smallest MMSE), hence is an ML & MMSE detector.

C. Numerical Results

We next present numerical results for the proposed scheme using the IEEE-4 bus system (Fig. 2). Note that in this case the state status is characterized by a 3-dimensional vector, i.e., $\mathbf{x} \in \mathbb{R}^3$ (the phase angle of bus 1 is taken as the reference). In Fig. 2, it is seen that there are eight measurements collected by meters, thus the topology is characterized by a 8-by-3 matrix, i.e., $\mathbf{H} \in \mathbb{R}^{8 \times 3}$.

Since the impedances of all links are known beforehand, we assume that they are of unit values without loss of generality. Here, instead of considering all possible forms of \mathbf{H} , we narrow down the candidate grid topologies to the outage scenarios. In particular, as given in (75), \mathbf{H}_0 represents the default topology matrix, and $\{\mathbf{H}_i, i = 1, 2, 3, 4\}$ correspond to the scenarios where the links $\{l_{1-2}, l_{2-3}, l_{3-4}, l_{4-1}\}$ (l_{i-j} denotes the link between bus i and bus j) break down, respectively.

We use the following distributions for the state vector \mathbf{x} under the hypotheses $\{\mathbf{H}_i\}$.

$$\begin{aligned} \mathbf{H}_0 : \mathbf{x} &\sim \mathcal{N}(\pi/5 \times \mathbf{1}, \pi^2/9 \times \mathbf{I}), & \mathbf{H}_1 : \mathbf{x} &\sim \mathcal{N}(2\pi/5 \times \mathbf{1}, \pi^2/16 \times \mathbf{I}), \\ \mathbf{H}_2 : \mathbf{x} &\sim \mathcal{N}(3\pi/5 \times \mathbf{1}, \pi^2/25 \times \mathbf{I}), & \mathbf{H}_3 : \mathbf{x} &\sim \mathcal{N}(4\pi/5 \times \mathbf{1}, \pi^2/36 \times \mathbf{I}), \\ \mathbf{H}_4 : \mathbf{x} &\sim \mathcal{N}(\pi \times \mathbf{1}, \pi^2/4 \times \mathbf{I}), \end{aligned} \tag{74}$$

where $a_i = 0.2$, $b_i = 0.8, \forall i$, $\mathbf{1}$ is the vector of ones and \mathbf{I} is the identity matrix. The measurements are contaminated by the white Gaussian noise $\mathbf{w}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$. The goal is to decide among the five candidate

grid topologies, and meanwhile, to estimate the state vector.

$$\begin{aligned}
 & \begin{matrix} & \theta_2 & \theta_3 & \theta_4 \\ P_1 & \begin{pmatrix} -1 & 0 & -1 \end{pmatrix} \\ P_{1-2} & \begin{pmatrix} -1 & 0 & 0 \end{pmatrix} \\ P_2 & \begin{pmatrix} 2 & -1 & 0 \end{pmatrix} \\ P_{2-3} & \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \\ P_3 & \begin{pmatrix} -1 & 2 & -1 \end{pmatrix} \\ P_{3-4} & \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \\ P_4 & \begin{pmatrix} 0 & -1 & 2 \end{pmatrix} \\ P_{4-1} & \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \end{matrix}, \quad \mathbf{H}_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \\
 \\
 \mathbf{H}_2 = \begin{pmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{H}_3 = \begin{pmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{H}_4 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (75)
 \end{aligned}$$

Since SPRT is not applicable in the multi-hypothesis case, we compare the proposed sequential joint detection and estimation (SJDE) scheme with the combination of maximum likelihood (ML) detector and MMSE estimator, equipped the stopping time given in (68). The ML detector uses the decision function

$$d_T = \arg \max_j a_j p_j \left(\{\mathbf{y}_t\}_{t=1}^T \right) \quad (76)$$

at the optimum stopping time presented in Corollary 3, hence is not a completely conventional scheme. Fig. 3 illustrates that SJDE [i.e., the hybrid ML & MMSE detector, given by (69)] significantly outperforms this combination [i.e., the conventional ML detector in (76)] in terms of the overall detection and estimation performance measured by the combined cost function, introduced in (66). We see that SJDE requires smaller average number of samples than ML & Est. to achieve the same target accuracy. Specifically, with small average sample size (i.e., stopping time), the improvement of SJDE is substantial.

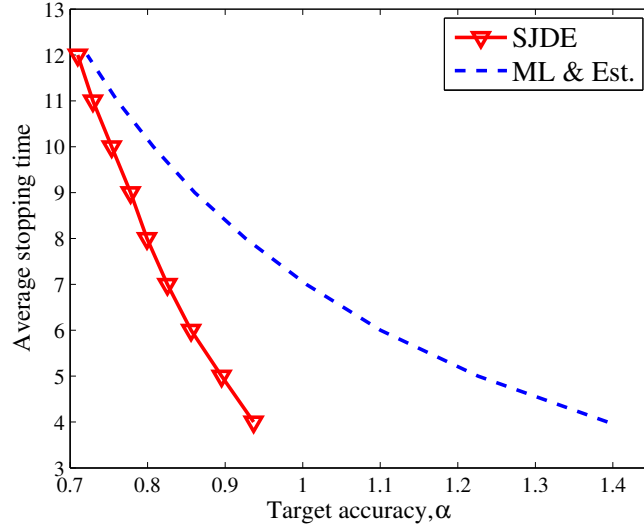


Fig. 3. Average stopping time vs. target accuracy level for SJDE and the combination of ML detector & MMSE estimator equipped with the stopping rule of SJDE.

This is because smaller sample size causes larger estimation cost Δ_T^j , which in turn emphasizes the advantage of the proposed detector over the conventional ML detector. In fact, in smart grid monitoring, the typical sample size is small since the system state evolves quickly, and thus there is limited time to estimate the current state.

VI. CONCLUSION

We have developed a general framework for optimum sequential joint detection (i.e., multi-hypothesis testing) and estimation, considering the problems in which simultaneous detection and estimation with minimal sample size is of interest. The proposed framework guarantees the best overall detection and estimation performance under a Bayesian setup while minimizing the sample size. The conventional separate treatment of the two subproblems has been shown to be strictly suboptimal since the optimum detector and estimators are strongly coupled with each other. We have also showed how the theoretical results, that are derived for a general model, apply to commonly used LQG models, including dynamic spectrum access in cognitive radio and state estimation in smart grid. We have supported the theoretical findings with numerical results.

APPENDIX A: PROOF OF LEMMA 1

If we find the estimators that minimize the cost function $\mathcal{C}(T, d_T, \{\hat{x}_T^i\})$ for any stopping time T and detector d_T , then, from (5), these estimators are the optimum estimators $\{\hat{x}_T^i\}$. Grouping the terms with

the same estimator in (3), we can write the optimum estimators as

$$\hat{\mathbf{x}}_T^i = \arg \min_{\hat{\mathbf{x}}} \sum_{j=0}^I b_{ji} \mathbb{E}_j [J(\hat{\mathbf{x}}, \mathbf{x}) \mathbb{1}_{\{d_T=i\}} | \mathcal{F}_T] \quad (77)$$

Using the likelihood ratio

$$\bar{L}_T^{ji} \triangleq \frac{p_j(\{\mathbf{y}_t\}_{t=1}^T, \mathbf{x} | \mathcal{F}_T)}{p_i(\{\mathbf{y}_t\}_{t=1}^T, \mathbf{x} | \mathcal{F}_T)} \quad (78)$$

we can write

$$\mathbb{E}_j [J(\hat{\mathbf{x}}, \mathbf{x}) \mathbb{1}_{\{d_T=i\}} | \mathcal{F}_T] = \mathbb{E}_i [\bar{L}_T^{ji} J(\hat{\mathbf{x}}, \mathbf{x}) \mathbb{1}_{\{d_T=i\}} | \mathcal{F}_T], \quad (79)$$

and accordingly

$$\hat{\mathbf{x}}_T^i = \arg \min_{\hat{\mathbf{x}}} \mathbb{E}_i \left[\sum_{j=0}^I b_{ji} \bar{L}_T^{ji} J(\hat{\mathbf{x}}, \mathbf{x}) \mathbb{1}_{\{d_T=i\}} | \mathcal{F}_T \right]. \quad (80)$$

To free the expectation from random T we first rewrite the above equation as

$$\hat{\mathbf{x}}_T^i = \arg \min_{\hat{\mathbf{x}}} \mathbb{E}_i \left[\sum_{t=0}^{\infty} \sum_{j=0}^I b_{ji} \bar{L}_t^{ji} J(\hat{\mathbf{x}}, \mathbf{x}) \mathbb{1}_{\{d_t=i\}} \mathbb{1}_{\{T=t\}} | \mathcal{F}_t \right], \quad (81)$$

then take $\mathbb{1}_{\{T=t\}}$ outside the expectation

$$\hat{\mathbf{x}}_T^i = \arg \min_{\hat{\mathbf{x}}} \sum_{t=0}^{\infty} \mathbb{E}_i \left[\sum_{j=0}^I b_{ji} \bar{L}_t^{ji} J(\hat{\mathbf{x}}, \mathbf{x}) \mathbb{1}_{\{d_t=i\}} | \mathcal{F}_t \right] \mathbb{1}_{\{T=t\}}, \quad (82)$$

as T is $\{\mathcal{F}_t\}$ -adapted, hence $\mathbb{1}_{\{T=t\}}$ is \mathcal{F}_t -measurable, i.e., deterministic given \mathcal{F}_t .

Since $\mathbb{E}_i[\cdot | \mathcal{F}_t] = \mathbb{E}_i[\mathbb{E}_i[\cdot | \mathcal{G}_t] | \mathcal{F}_t]$, we write

$$\hat{\mathbf{x}}_T^i = \arg \min_{\hat{\mathbf{x}}} \sum_{t=0}^{\infty} \mathbb{E}_i \left[\sum_{j=0}^I b_{ji} \mathbb{E}_i \left[\bar{L}_t^{ji} J(\hat{\mathbf{x}}, \mathbf{x}) \mathbb{1}_{\{d_t=i\}} | \mathcal{G}_t \right] | \mathcal{F}_t \right] \mathbb{1}_{\{T=t\}}. \quad (83)$$

Note that d_t is \mathcal{G}_t -measurable, i.e., a feasible detector is a function of the observations only, hence deterministic given \mathcal{G}_t . Then, we have

$$\hat{\mathbf{x}}_T^i = \arg \min_{\hat{\mathbf{x}}} \sum_{t=0}^{\infty} \mathbb{E}_i \left[\sum_{j=0}^I b_{ji} \mathbb{E}_i \left[\bar{L}_t^{ji} J(\hat{\mathbf{x}}, \mathbf{x}) | \mathcal{G}_t \right] \mathbb{1}_{\{d_t=i\}} | \mathcal{F}_t \right] \mathbb{1}_{\{T=t\}}, \quad (84)$$

which reduces to

$$\hat{\mathbf{x}}_T^i = \arg \min_{\hat{\mathbf{x}}} \sum_{t=0}^{\infty} \sum_{j=0}^I b_{ji} \mathbb{E}_i \left[\bar{L}_t^{ji} J(\hat{\mathbf{x}}, \mathbf{x}) | \mathcal{G}_t \right] \mathbb{1}_{\{T=t\}}. \quad (85)$$

Expand the likelihood ratio \bar{L}_t^{ji} as

$$\begin{aligned} \bar{L}_t^{ji} &= \frac{p_j(\{\mathbf{y}_s\}_{s=1}^t, \mathbf{x} | \mathcal{F}_t)}{p_i(\{\mathbf{y}_s\}_{s=1}^t, \mathbf{x} | \mathcal{F}_t)} \\ &= \frac{p_j(\{\mathbf{y}_s\}_{s=1}^t | \mathcal{F}_t)}{p_i(\{\mathbf{y}_s\}_{s=1}^t | \mathcal{F}_t)} \frac{p_j(\mathbf{x} | \{\mathbf{y}_s\}_{s=1}^t, \mathcal{F}_t)}{p_i(\mathbf{x} | \{\mathbf{y}_s\}_{s=1}^t, \mathcal{F}_t)} \\ &= \frac{p_j(\{\mathbf{y}_s\}_{s=1}^t | \mathcal{F}_t)}{p_i(\{\mathbf{y}_s\}_{s=1}^t | \mathcal{F}_t)} \frac{p_j(\mathbf{x} | \mathcal{G}_t)}{p_i(\mathbf{x} | \mathcal{G}_t)}, \end{aligned} \quad (86)$$

and denote the first term above with

$$L_t^{ji} = \frac{p_j(\{\mathbf{y}_s\}_{s=1}^t | \mathcal{F}_t)}{p_i(\{\mathbf{y}_s\}_{s=1}^t | \mathcal{F}_t)}, \quad (87)$$

which is also a likelihood ratio. Given \mathcal{G}_t , L_t^{ji} is deterministic, hence in (85), within \bar{L}_t^{ji} , only $\frac{p_j(\mathbf{x} | \mathcal{G}_t)}{p_i(\mathbf{x} | \mathcal{G}_t)}$ remains inside the expectation. Since

$$\mathbb{E}_i \left[\frac{p_j(\mathbf{x} | \mathcal{G}_t)}{p_i(\mathbf{x} | \mathcal{G}_t)} J(\hat{\mathbf{x}}, \mathbf{x}) | \mathcal{G}_t \right] = \mathbb{E}_j [J(\hat{\mathbf{x}}, \mathbf{x}) | \mathcal{G}_t], \quad (88)$$

we rewrite (85) as

$$\hat{\mathbf{x}}_T^i = \arg \min_{\hat{\mathbf{x}}} \sum_{t=0}^{\infty} \sum_{j=0}^I b_{ji} L_t^{ji} \mathbb{E}_j [J(\hat{\mathbf{x}}, \mathbf{x}) | \mathcal{G}_t] \mathbb{1}_{\{T=t\}}. \quad (89)$$

Define a new probability distribution

$$\bar{p}_t^i(\mathbf{x} | \mathcal{G}_t) \triangleq \frac{\sum_{j=0}^I b_{ji} L_t^{ji} p_j(\mathbf{x} | \mathcal{G}_t)}{\sum_{j=0}^I b_{ji} L_t^{ji}}.$$

We are, in fact, searching for an estimator that minimizes $\bar{\mathbb{E}}_i [J(\hat{\mathbf{x}}, \mathbf{x}) | \mathcal{G}_T]$ under $\bar{p}_t^i(\mathbf{x} | \mathcal{G}_t)$, i.e.,

$$\hat{\mathbf{x}}_T^i = \arg \min_{\hat{\mathbf{x}}} \bar{\mathbb{E}}_i [J(\hat{\mathbf{x}}, \mathbf{x}) | \mathcal{G}_T], \quad (90)$$

which proves (6). The MMSE estimator, for which $J(\hat{\mathbf{x}}, \mathbf{x}) = \|\hat{\mathbf{x}} - \mathbf{x}\|^2$, is given by the conditional mean $\bar{\mathbb{E}}_i[\mathbf{x} | \mathcal{G}_T]$, hence the result in (10), concluding the proof.

APPENDIX B: PROOF OF LEMMA 2

Reorganizing the terms in (12) we get

$$d_T = \arg \min_{d_T} \sum_{i=0}^I \mathbb{E}_i \left[a_i \mathbb{1}_{\{d_T \neq i\}} + b_{ii} \Delta_T^{ii} \mathbb{1}_{\{d_T = i\}} + \sum_{j=0, j \neq i}^I b_{ij} \Delta_T^{ij} \mathbb{1}_{\{d_T = j\}} | \mathcal{F}_T \right]. \quad (91)$$

Using $\mathbb{1}_{\{d_T = i\}} = 1 - \mathbb{1}_{\{d_T \neq i\}}$, it becomes

$$\begin{aligned} d_T &= \arg \min_{d_T} \sum_{i=0}^I \mathbb{E}_i \left[a_i \mathbb{1}_{\{d_T \neq i\}} - b_{ii} \Delta_T^{ii} \mathbb{1}_{\{d_T \neq i\}} + \sum_{j=0, j \neq i}^I b_{ij} \Delta_T^{ij} \mathbb{1}_{\{d_T = j\}} | \mathcal{F}_T \right] + b_{ii} \mathbb{E}_i [\Delta_T^{ii} | \mathcal{F}_T] \\ &= \arg \min_{d_T} \sum_{i=0}^I \mathbb{E}_i \left[\left(a_i - b_{ii} \Delta_T^{ii} + \sum_{j=0, j \neq i}^I b_{ij} \Delta_T^{ij} \mathbb{1}_{\{d_T = j\}} \right) \mathbb{1}_{\{d_T \neq i\}} | \mathcal{F}_T \right] \\ &= \arg \min_{d_T} \sum_{i=0}^I \int \cdots \int_{\mathcal{R}_i(d_T)} \left(a_i - b_{ii} \Delta_T^{ii} + \sum_{j=0, j \neq i}^I b_{ij} \Delta_T^{ij} \mathbb{1}_{\{d_T = j\}} \right) p_i(\{\mathbf{y}_t\}_{t=1}^T | \mathcal{F}_T) d\mathbf{y}_1 \cdots d\mathbf{y}_T, \end{aligned} \quad (92)$$

where $\mathcal{R}_i(d_T) \in \mathbb{R}^{M \times T}$ is the subspace where H_i is rejected, i.e., $d_T \neq i$. When H_i is selected given an observation set $\{\mathbf{y}_t\}_{t=1}^T$, the corresponding cost $\left(a_i - b_{ii} \Delta_T^{ii} + \sum_{j=0, j \neq i}^I b_{ij} \Delta_T^{ij} \mathbb{1}_{\{d_T = j\}} \right) p_i(\{\mathbf{y}_t\}_{t=1}^T | \mathcal{F}_T)$

is omitted in the risk calculation, i.e., the cost

$$\sum_{j=0, j \neq i}^I \left(a_j - b_{jj} \Delta_T^{jj} + b_{ji} \Delta_T^{ji} \right) p_j \left(\{\mathbf{y}_t\}_{t=1}^T | \mathcal{F}_T \right) \quad (93)$$

is included in the average risk. Hence, the optimum detector is the minimizer of this conditional risk, as shown in (16).

For the special case of binary hypothesis testing, $(a_1 + b_{10} \Delta_T^{10} - b_{11} \Delta_T^{11}) p_1(\{\mathbf{y}_t\}_{t=1}^T | \mathcal{F}_T)$ gives the conditional risk when H_0 is decided. Similarly, it is given by $(a_0 + b_{01} \Delta_T^{01} - b_{00} \Delta_T^{00}) p_0(\{\mathbf{y}_t\}_{t=1}^T | \mathcal{F}_T)$ when H_1 is decided. Therefore, the optimum detector chooses H_1 if

$$(a_0 + b_{01} \Delta_T^{01} - b_{00} \Delta_T^{00}) p_0(\{\mathbf{y}_t\}_{t=1}^T | \mathcal{F}_T) \leq (a_1 + b_{10} \Delta_T^{10} - b_{11} \Delta_T^{11}) p_1(\{\mathbf{y}_t\}_{t=1}^T | \mathcal{F}_T), \quad (94)$$

and chooses H_0 otherwise.

APPENDIX C: PROOF OF PROPOSITION 1

We start by deriving the joint distribution density function of $\{\mathbf{y}_s\}_{s=1}^t$ and \mathbf{x} as follows:

$$\begin{aligned} p_i(\{\mathbf{y}_s\}_{s=1}^t, \mathbf{x} | \mathcal{F}_t) &= p_i(\{\mathbf{y}_s\}_{s=1}^t | \mathbf{x}, \mathcal{F}_t) p_i(\mathbf{x}) \\ &= \frac{\exp\left(-\frac{1}{2\sigma^2} \sum_{s=1}^t \|\mathbf{y}_s - \mathbf{H}_s \mathbf{x}\|^2\right)}{(2\pi)^{mt/2} \sigma^{mt}} \frac{\exp\left(-\frac{1}{2} \|\mathbf{x} - \boldsymbol{\mu}_i\|_{\boldsymbol{\Sigma}_i^{-1}}^2\right)}{(2\pi)^{n/2} |\boldsymbol{\Sigma}_i|^{1/2}} \\ &= \exp\left(-\frac{1}{2} \left(\sum_{s=1}^t \frac{\|\mathbf{y}_s\|^2}{\sigma^2} + \|\boldsymbol{\mu}_i\|_{\boldsymbol{\Sigma}_i^{-1}}^2 - \left\| \frac{\mathbf{v}_t}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\mu}_i \right\|_{\left(\frac{\mathbf{U}_t}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1}\right)^{-1}}^2 \right)\right) \\ &\quad \exp\left(-\frac{1}{2} \left\| \mathbf{x} - \left(\frac{\mathbf{U}_t}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1}\right)^{-1} \left(\frac{\mathbf{v}_t}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\mu}_i\right) \right\|_{\frac{\mathbf{U}_t}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1}}^2\right) \frac{1}{(2\pi)^{mt/2+n/2} \sigma^{mt} |\boldsymbol{\Sigma}_i|^{1/2}}, \end{aligned} \quad (95)$$

where $\mathbf{U}_t = \sum_{s=1}^t \mathbf{H}_s' \mathbf{H}_s$ and $\mathbf{v}_t = \sum_{s=1}^t \mathbf{H}_s' \mathbf{y}_s$. Since the Gaussian prior $p_i(\mathbf{x})$ is conjugate to the Gaussian likelihood function $p_i(\{\mathbf{y}_s\}_{s=1}^t | \mathbf{x}, \mathcal{F}_t)$, the posterior distribution $p_i(\mathbf{x} | \{\mathbf{y}_s\}_{s=1}^t, \mathcal{F}_t)$ is also Gaussian. Furthermore, due to

$$p_i(\mathbf{x} | \mathcal{G}_t) = \frac{p_i(\{\mathbf{y}_s\}_{s=1}^t, \mathbf{x} | \mathcal{F}_t)}{p_i(\{\mathbf{y}_s\}_{s=1}^t | \mathcal{F}_t)}, \quad (96)$$

we can read off the mean and variance of $\mathbf{x} | \mathcal{G}_t$ from the second exponent in (95), which is the only term involving \mathbf{x} in $p_i(\mathbf{x} | \mathcal{G}_t)$, and arrive at

$$\mathbf{x} | \mathcal{G}_t \sim \mathcal{N}\left(\underbrace{\left(\frac{\mathbf{U}_t}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1}\right)^{-1} \left(\frac{\mathbf{v}_t}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\mu}_i\right)}_{\mathbf{E}_i[\mathbf{x} | \mathcal{G}_t]}, \underbrace{\left(\frac{\mathbf{U}_t}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1}\right)^{-1}}_{\mathbf{Cov}_i[\mathbf{x} | \mathcal{G}_t]}\right), \quad (97)$$

which proves (37). Moreover, (14) and (97) give (38). Finally, the likelihood function of $\{\mathbf{y}_s\}_{s=1}^t$ is computed as

$$\begin{aligned} p_i(\{\mathbf{y}_s\}_{s=1}^t | \mathcal{F}_t) &= \frac{p_i(\{\mathbf{y}_s\}_{s=1}^t, \mathbf{x} | \mathcal{F}_t)}{p_i(\mathbf{x} | \{\mathbf{y}_s\}_{s=1}^t, \mathcal{F}_t)} \\ &= \frac{\exp \left[-\frac{1}{2} \left(\sum_{s=1}^t \frac{\|\mathbf{y}_s\|^2}{\sigma^2} + \|\boldsymbol{\mu}_i\|_{\boldsymbol{\Sigma}_i^{-1}}^2 - \left\| \frac{\mathbf{v}_t}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\mu}_i \right\|_{\left(\frac{\mathbf{U}_t}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1} \right)^{-1}}^2 \right) \right]}{(2\pi)^{mt/2} \sigma^{mt} |\boldsymbol{\Sigma}_i|^{1/2} \left| \frac{\mathbf{U}_t}{\sigma^2} + \boldsymbol{\Sigma}_i^{-1} \right|^{1/2}}. \end{aligned} \quad (98)$$

The likelihood ratio L_T in (39) follows from (98), concluding the proof.

APPENDIX D: PROOF OF PROPOSITION 2

Since \mathbf{H}_t is diagonal and both \mathbf{x} and \mathbf{w}_t have independent entries, the linear system model (35) can be decomposed into N sub-systems, i.e., $y_{t,n} = h_{t,n}x_n + w_n$, $n = 1, 2, \dots, N$, which are independent from each other. Then the posterior distribution is a scalar version of (97) for each local parameter x_n , i.e.,

$$x_n | \mathcal{G}_t \sim \mathcal{N} \left(\frac{\frac{v_{t,n}}{\sigma^2} + \frac{\mu_{i,n}}{\rho_{i,n}^2}}{\frac{u_{t,n}}{\sigma^2} + \frac{1}{\rho_{i,n}^2}}, \frac{1}{\frac{u_{t,n}}{\sigma^2} + \frac{1}{\rho_{i,n}^2}} \right), \quad (99)$$

proving (43). Moreover, due to spatial independence, we have

$$p_i(\{\mathbf{y}_s\}_{s=1}^t | \mathcal{F}_t) = \prod_{n=1}^N p_i(\{y_{s,n}\}_{s=1}^t | \mathcal{F}_t^n), \quad (100)$$

where $p_i(\{y_{s,n}\}_{s=1}^t | \mathcal{F}_t^n)$ is given by the scalar version of (98), i.e.,

$$p_i(\{y_{s,n}\}_{s=1}^t | \mathcal{F}_t^n) = \frac{\exp \left[-\frac{1}{2} \left(\sum_{s=1}^t \frac{y_{s,n}^2}{\sigma^2} + \frac{\mu_{i,n}^2}{\rho_{i,n}^2} - \frac{\left(\frac{v_{t,n}}{\sigma^2} + \frac{\mu_{i,n}}{\rho_{i,n}^2} \right)^2}{\frac{u_{t,n}}{\sigma^2} + \frac{1}{\rho_{i,n}^2}} \right) \right]}{(2\pi)^{t/2} \sigma^t \rho_{i,n} \sqrt{\frac{u_{t,n}}{\sigma^2} + \frac{1}{\rho_{i,n}^2}}}. \quad (101)$$

The global likelihood ratio is given by the product of the local ones, i.e., $L_t = \prod_{n=1}^N L_t^n$, where, from (101),

$$L_t^n = \frac{p_1(\{y_{s,n}\}_{s=1}^t | \mathcal{F}_t^n)}{p_0(\{y_{s,n}\}_{s=1}^t | \mathcal{F}_t^n)} \quad (102)$$

is written as in (45). From (13),

$$\Delta_T^{ij} = \sum_{n=1}^N \text{Var}_i[x_n | \mathcal{G}_T^n] + \delta_T^{ij} \|\mathbf{E}_0[\mathbf{x} | \mathcal{G}_T] - \mathbf{E}_1[\mathbf{x} | \mathcal{G}_T]\|^2, \quad (103)$$

which, together with (99), gives (44), concluding the proof.

APPENDIX E: PROOF OF COROLLARY 2

Substituting $I = 1$ and $b_{01} = 0$ into (21) we write the optimum estimator as

$$\hat{\mathbf{x}}_T = \mathbf{E}_1[\mathbf{x} | \mathcal{G}_T], \quad (104)$$

which is used only when H_1 is decided. Since the independent LQG model (i.e., diagonal \mathbf{H}_t and $\mathbf{\Sigma}$) is used in the problem formulation, we can borrow, from Proposition 2, the result for $E_1[x|\mathcal{G}_T]$, given by (43), to write (61).

From (13), we write

$$\Delta_T^{11} = \text{Tr}(\text{Cov}_1[x|\mathcal{G}_T]) \quad \text{and} \quad \Delta_T^{10} = \text{Tr}(\text{Cov}_1[x|\mathcal{G}_T]) + \|\hat{\mathbf{x}}_T\|^2, \quad (105)$$

where we used $\hat{\mathbf{x}}_T^1 = E_1[x|\mathcal{G}_T]$ and $\hat{\mathbf{x}}_T^0 = 0$. Then, in the optimum detector expression given by (16), on the right side we only have a_0 since $b_{01} = b_{00} = 0$; and on the left side we have $L_T(a_1 + b_1\|\hat{\mathbf{x}}_T\|^2)$ since $b_{10} = b_{11} = b_1$, resulting in (60).

Similarly, using $b_{01} = b_{00} = 0$ and $b_{10} = b_{11} = b_1$ in (22), the optimal cost is given by

$$\mathcal{C}_t = E_0[a_0 \mathbb{1}_{\{d_t=1\}}|\mathcal{F}_t] + E_1[b_1 \Delta_t^{11} \mathbb{1}_{\{d_t=1\}} + (a_1 + b_1 \Delta_t^{10}) \mathbb{1}_{\{d_t=0\}}|\mathcal{F}_t]. \quad (106)$$

Using $\mathbb{1}_{\{d_t=0\}} = 1 - \mathbb{1}_{\{d_t=1\}}$, $\Delta_t^{11} - \Delta_t^{10} = -\|\hat{\mathbf{x}}_t\|^2$, and combining the expectations under E_0 via L_t we write

$$\mathcal{C}_t = E_0[\{a_0 - (a_1 + b_1\|\hat{\mathbf{x}}_t\|^2) L_t\} \mathbb{1}_{\{d_t=1\}} + b_1 L_t \|\hat{\mathbf{x}}_t\|^2 |\mathcal{F}_t] + a_1 + b_1 \sum_{j=1}^2 \sum_{k=1}^K \text{Var}_1[x_{jk}|\mathcal{G}_t]. \quad (107)$$

Since $\mathbb{1}_{\{d_t=1\}}$ passes only the negative values of $a_0 - (a_1 + b_1\|\hat{\mathbf{x}}_t\|^2) L_t$, we obtain (62). The MMSE (i.e., sum of posterior variances) expression follows from (99). For the likelihood ratio, due to independence, we have

$$L_t = \prod_{j=1}^2 \prod_{k=1}^K L_t^{jk} \quad \text{where} \quad L_t^{jk} = \frac{p_1(\{y_{s,jk}\}_{s=1}^t | \mathcal{F}_t^{jk})}{p_0(\{y_{s,jk}\}_{s=1}^t | \mathcal{F}_t^{jk})} \quad (108)$$

is the local likelihood ratio for the channel between PU j and SU k . The likelihood $p_1(\{y_{s,jk}\}_{s=1}^t | \mathcal{F}_t^{jk})$ is given by (101); and

$$p_0(\{y_{s,jk}\}_{s=1}^t | \mathcal{F}_t^{jk}) = \frac{\exp\left(-\frac{1}{2} \sum_{s=1}^t \frac{y_{s,jk}^2}{\sigma^2}\right)}{(2\pi)^{t/2} \sigma^t} \quad (109)$$

since the received signal under H_0 is white Gaussian noise. Hence, L_t is written as in (63).

APPENDIX F: PROOF OF COROLLARY 3

Since separated detection and estimation costs (cf. Section II-C) are used in the problem formulation, from Corollary 1, when H_i is decided, the optimum estimator under H_i is used. As a result, the posterior estimation cost Δ_T^i corresponds to the MSE, which, from (38), is given by (72). For the LQG model assumed in (64)–(66), the optimum estimator is the MMSE estimator, and, from (37), written as in (70). The optimum detector also follows from Corollary 1. Since Δ_t^i is deterministic, from (31), the optimal cost \mathcal{C}_t is given by (71). Finally, from (98), $p_j(\{\mathbf{y}_t\}_{t=1}^T)$ is given by (73).

REFERENCES

- [1] Y. Yilmaz, Z. Guo, and X. Wang, "Sequential Joint Spectrum Sensing and Channel Estimation for Dynamic Spectrum Access," *IEEE J. Sel. Areas Commun.*, vol. 32, no. 11, pp. 2000–2012, Nov. 2014.
- [2] J. Chen, Y. Zhao, A. Goldsmith, and H. V. Poor, "Optimal Joint Detection and Estimation in Linear Models," in *Proc. IEEE 52nd Annual Conference on Decision and Control (CDC)*, pp. 4416–4421, Dec. 2013.
- [3] B.-N. Vo, B.-T. Vo, N.-T. Pham and D. Suter, "Joint detection and estimation of multiple objects from image observations," *IEEE Trans. Signal Process.*, vol. 58, no. 10, pp. 5129–5141, Oct. 2010.
- [4] A. Tajer, G.H. Jajamovich, X. Wang, and G.V. Moustakides, "Optimal Joint Target Detection and Parameter Estimation by MIMO Radar," *IEEE J. Sel. Topics Signal Process.*, vol. 4, no. 1, pp. 127–145, Feb. 2010.
- [5] G.H. Jajamovich, A. Tajer, and X. Wang, "Minimax-Optimal Hypothesis Testing With Estimation-Dependent Costs," *IEEE Trans. Signal Process.*, vol. 60, no. 12, pp. 6151–6165, Dec. 2012.
- [6] D. Ciuonzo, A. De Maio, and D. Orlando, "A Unifying Framework for Adaptive Radar Detection in Homogeneous Plus Structured Interference–Part II: Detectors Design," *IEEE Trans. Signal Process.*, vol. 64, no. 11, pp. 2907–2919, June 2016.
- [7] S.M. Kay, *Fundamentals of Statistical Signal Processing, Volume 2: Detection Theory*, Prentice Hall, Englewood Cliffs, NJ, 1998.
- [8] D. Middleton, and R. Esposito, "Simultaneous optimum detection and estimation of signals in noise", *IEEE Trans. Inf. Theory*, vol. 14, no. 3, pp. 434–444, May 1968.
- [9] Y. Yilmaz, G.V. Moustakides, and X. Wang, "Sequential Joint Detection and Estimation," *SIAM Theory Probab. Appl.*, vol. 59, no. 3, pp. 452–465, 2015 (also in *Teor. Veroyatnost. i Primenen.*, vol. 59, no. 3, pp. 562–578, 2014)
- [10] O. Zeitouni, J. Ziv, and N. Merhav, "When is the generalized likelihood ratio test optimal?", *IEEE Trans. Inf. Theory*, vol. 38, no. 5, pp. 1597–1602, Sept. 1992.
- [11] G.V. Moustakides, "Optimum Joint Detection and Estimation," in *Proc. IEEE International Symposium on Information Theory (ISIT)*, pp. 2984–2988 July 2011,
- [12] A. Fredriksen, D. Middleton, and D. Vandelinde, "Simultaneous Signal Detection and Estimation Under Multiple Hypotheses", *IEEE Trans. Inf. Theory*, vol. 18, no. 5, pp. 607–614, Sept. 1972.
- [13] T.G. Birdsall, and J.O. Gobien, "Sufficient Statistics and Reproducing Densities in Simultaneous Sequential Detection and Estimation", *IEEE Trans. Inf. Theory*, vol. 19, no. 6, pp. 760–768, Nov. 1973.
- [14] B. Baygün, and A.O. Hero III, "Optimal Simultaneous Detection and Estimation Under a False Alarm Constraint", *IEEE Trans. Inf. Theory*, vol. 41, no. 3, pp. 688–703, May 1995.
- [15] G.V. Moustakides, G.H. Jajamovich, A. Tajer, and X. Wang, "Joint Detection and Estimation: Optimum Tests and Applications," *IEEE Trans. Inf. Theory*, vol. 58, no. 7, pp. 4215–4229, July 2012.
- [16] S. Li, and X. Wang "Optimal Joint Detection and Estimation Based on Decision-Dependent Bayesian Cost," *IEEE Trans. Signal Process.*, vol. 64, no. 10, pp. 2573–2586, May 2016.
- [17] S. Li, and X. Wang "Joint Composite Detection and Bayesian Estimation: A Neyman-Pearson Approach," in *Proc. of 3rd Global Conference on Signal and Information Processing*, pp. 453–457, Orlando, FL, 2015.
- [18] A. Wald, *Sequential Analysis*, Wiley, New York, NY, 1947.
- [19] Z. Govindarajulu, *Sequential Statistics*, World Scientific Publishing, Hackensack, NJ, 2004.
- [20] S. Boyd, and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, New York, NY, 2009.

- [21] H. Zhang, J.M. Moura, and B. Krogh, "Dynamic Field Estimation Using Wireless Sensor Networks: Tradeoffs Between Estimation Error and Communication Cost," *IEEE Trans. Signal Process.*, vol. 57, no. 6, pp. 2383–2395, June 2009.
- [22] Y. Yilmaz, G.V. Moustakides, and X. Wang, "Sequential and Decentralized Estimation of Linear-Regression Parameters in Wireless Sensor Networks," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 52, no. 1, pp. 288–306, Feb. 2016.
- [23] G. Mateos, J.A. Bazerque, and G.B. Giannakis, "Distributed Sparse Linear Regression," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 58, no. 10, pp. 5262–5276, Oct. 2010.
- [24] B. Efron, and D.V. Hinkley, "Assessing the accuracy of the maximum likelihood estimator: Observed versus expected Fisher information," *Biometrika*, vol. 65, no. 3, pp. 457–487, 1978.
- [25] A. Birnbaum, "On the foundations of statistical inference," *Journal of the American Statistical Association*, vol. 57, no. 298, pp. 269–326, 1962.
- [26] B.K. Ghosh, "On the attainment of the Cramer-Rao bound in the sequential case," *Seq. Anal.*, vol. 6, no. 3, pp. 267–288, 1987.
- [27] B.K. Ghosh, and P.K. Sen, *Handbook of Sequential Analysis*, Marcel Dekker, New York, NY, 1991.
- [28] P. Grambsch, "Sequential sampling based on the observed Fisher information to guarantee the accuracy of the maximum likelihood estimator," *Ann. Statist.*, vol. 11, no. 1, pp. 68–77, 1983.
- [29] G. Fellouris, "Asymptotically optimal parameter estimation under communication constraints," *Ann. Statist.*, vol. 40, no. 4, pp. 2239–2265, Aug. 2012.
- [30] Y. Yilmaz, and X. Wang, "Sequential Decentralized Parameter Estimation under Randomly Observed Fisher Information," *IEEE Trans. Inf. Theory*, vol. 60, no. 2, pp. 1281–1300, Feb. 2014.
- [31] H.L. Van Trees, and K.L. Bell, *Detection Estimation and Modulation Theory, Part I (2nd Edition)*, Wiley, Somerset, NJ, 2013.
- [32] Q. Zhao and B. M. Sadler, "A survey of dynamic spectrum access: Signal processing, networking, and regulatory policy," *IEEE Signal Processing Mag.*, vol. 24, no. 3, pp. 79–89, May 2007.
- [33] Y. C. Liang, Y. Zeng, E. C. Y. Peh, and A. T. Hoang, "Sensing-throughput tradeoff for cognitive radio networks," *IEEE Trans. Wireless Commun.*, vol. 7, no. 4, pp. 1326–1337, Apr. 2008.
- [34] Y. Chen, Q. Zhao, and A. Swami, "Joint design and separation principle for opportunistic spectrum access in the presence of sensing errors," *IEEE Trans. Inf. Theory*, vol. 54, no. 5, pp. 2053–2071, May 2008.
- [35] X. Kang, Y. C. Liang, A. Nallanathan, H. K. Garg, and R. Zhang, "Optimal power allocation for fading channels in cognitive radio networks: ergodic capacity and outage capacity," *IEEE Trans. Wireless Commun.*, vol. 8, no. 2, pp. 940–950, Feb. 2009.
- [36] L. Musavian and S. Aissa, "Capacity and power allocation for spectrum sharing communications in fading channels," *IEEE Trans. Wireless Commun.*, vol. 8, no. 1, pp. 148–156, 2009.
- [37] X. Kang, Y. C. Liang, H. K. Garg, and L. Zhang, "Sensing-based spectrum sharing in cognitive radio networks," *IEEE Trans. Veh. Technol.*, vol. 58, no. 8, pp. 4649–4654, Oct. 2009.
- [38] Z. Chen, X. Wang, and X. Zhang, "Continuous power allocation strategies for sensing-based multiband spectrum sharing," *IEEE J. Sel. Areas Commun.*, vol. 31, no. 11, pp. 2409–2419, Nov. 2013.
- [39] Y. Li, "Pilot-symbol-aided channel estimation for OFDM in wireless systems," *IEEE Trans. Veh. Technol.*, vol. 49, no. 4, pp. 1207–1215, July 2000.
- [40] A. Sahai, R. Tandra, S. M. Mishra, and N. Hoven, "Fundamental design tradeoffs in cognitive radio systems," in *Proc. of Int. Workshop on Technology and Policy for Accessing Spectrum*, Aug. 2006.

- [41] Y. Yilmaz, G.V. Moustakides, and X. Wang, “Cooperative sequential spectrum sensing based on level-triggered sampling,” *IEEE Trans. Signal Process.*, vol. 60, no. 9, pp. 4509–4524, Sep. 2012.
- [42] Y. Huang, S. Werner, J. Huang, N. Kashyap, and V. Gupta “State Estimation in Electric Power Grids: Meeting New Challenges Presented by the Requirements of the Future Grid,” *IEEE Signal Processing Mag.*, vol.29, no. 5, pp. 33–43, Sept. 2012.
- [43] Y. Huang, H. Li, K.A. Campbell, and Z. Han “Defending False Data Injection Attack On Smart Grid Network Using Adaptive CUSUM Test,” *45th Annual Conference on Information Sciences and Systems (CISS)*, 2011
- [44] Y. Huang, M. Esmalifalak, H. Li, K.A. Campbell, and Z. Han “Adaptive Quickest Estimation Algorithm for Smart Grid Network Topology Error,” *IEEE Syst. J.*, vol. 8, no. 2, pp. 430–440, 2014.
- [45] Y. Zhao, A. Goldsmith, and H. V. Poor “On PMU location selection for line outage detection in wide-area transmission networks,” *IEEE Power and Energy Society General Meeting*, 2012
- [46] Y. Zhao, R. Sevlain, R. Rajagopal, A. Goldsmith, and H. V. Poor “Outage Detection in Power Distribution Networks with Optimally-Deployed Power Flow Sensors,” *IEEE Power and Energy Society General Meeting*, 2013



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